

# Power multiples in binary recurrence sequences: an approach by congruences

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## Abstract

We introduce an elementary congruence-based procedure to look for  $q$ -th power multiples in arbitrary binary recurrence sequences ( $q \geq 3$ ). The procedure allows to prove that no such multiples exist in many instances.  
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## 1 Introduction and result

Let  $u, v, A, B \in \mathbb{Z}$ . The ( $\mathbb{Z}$ -valued) binary recurrence sequence with initial values  $u, v$  and coefficients  $A, B$  is the sequence  $\{G_n\}_{n \geq 0}$  defined recursively as

$$G_0 = u, \quad G_1 = v, \quad G_{n+2} = AG_{n+1} + BG_n \text{ for all } n \geq 0. \quad (1)$$

The discriminant of the sequence (1) is the integer  $\Delta = A^2 + 4B \neq 0$ . An equivalent description is

$$\begin{pmatrix} G_{n+2} \\ G_{n+1} \end{pmatrix} = \begin{pmatrix} A & B \\ 1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix}, \quad (2)$$

i.e.  $\begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} A & B \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} G_1 \\ G_0 \end{pmatrix}$ , for all  $n \geq 0$ . Let  $K$  be the smallest extension of  $\mathbb{Q}$  containing the eigenvalues  $\{\lambda_1, \lambda_2\}$  of the matrix  $\begin{pmatrix} A & B \\ 1 & 0 \end{pmatrix}$  and denote  $\mathcal{O}_K$  its ring of integers. Either  $K = \mathbb{Q}$  or  $K$  is quadratic,  $K = \mathbb{Q}(\sqrt{\Delta})$ , and in the latter case write  $\text{Gal}(K/\mathbb{Q}) = \langle \tau \rangle$ . The sequence (1) is called non-degenerate if  $\lambda_1/\lambda_2$  is not a root of 1. Also, if  $\lambda_1 \neq \lambda_2$  the sequence is a generalized power sum with constant coefficients, namely

$$G_n = g_1 \lambda_1^n + g_2 \lambda_2^n, \quad \text{where } g_1 = \frac{G_1 - \lambda_2 G_0}{\lambda_1 - \lambda_2}, \quad g_2 = \frac{\lambda_1 G_0 - G_1}{\lambda_1 - \lambda_2}.$$

A sequence with values in  $\mathbb{Z}$  can be “followed” looking for integers with special interesting arithmetic properties (Ribenoim [5] likens this to picking

wild flowers during a walk in the countryside). In this note we deal with the equation

$$G_n = kx^q \quad (3)$$

where  $0 \neq k \in \mathbb{Z}$  is a fixed constant and  $q \geq 3$ . As usual, we may and shall assume that  $q$  is a prime number.

By relating it to Baker's theory of linear forms in logarithms, Pethő [4] and Shorey and Stewart [6] proved independently that (3) has, under some mild conditions on the sequence, only finitely many solutions  $(n, G_n, x, q)$ . Pethő's precise version of the result is the following.

**Theorem 1.1.** *Let  $\{G_n\}$  be a binary recurrence sequence with coprime non-zero coefficients  $A$  and  $B$  such that  $(G_0, G_1) \neq (0, 0)$ ,  $A^2 \neq -jB$  for  $j \in \{1, 2, 3, 4\}$  and  $G_1^2 - AG_0G_1 - BG_0^2 \neq 0$ . Let  $\mathcal{P}$  be a finite set of primes and let  $\mathcal{S}$  be the set of integers divisible only by primes in  $\mathcal{P}$ . Then, there exists an effective constant  $C = C(A, B, G_0, G_1, \mathcal{P})$  such that if  $G_n = kx^q$  with  $k \in \mathcal{S}$  and  $|x| > 1$  then  $\max(n, |G_n|, |x|, q) < C$ .*

**Remark 1.2.** *When the sequence  $\{G_n\}$  is non-degenerate and  $k$  is any fixed integer, the finiteness of the number of solutions of  $G_n = k$  (i.e. the  $x$ -trivial solutions of (3)) follows from the Skolem-Mahler-Lech theorem, [3, §2.1], which is independent of Baker's theory.*

Although theorem 1.1 reduces in principle the problem of finding all the solutions of (3) to a finite amount of computations, from a practical point of view the possibility of using brute force is illusory since the constant  $C$  is huge. Following the steps of the proof of theorem 1.1 in the arguably simplest case of the Fibonacci sequence  $\{F_n\}$  (obtained for  $u = 0$ ,  $v = 1$ ,  $A = B = 1$ ) the first author [1] found that for a solution of (3) with  $k = 1$  the bounds are  $q \leq 192^{1203}$ , and  $|x| \leq e^{5^{80(4q^l+1)(4q^l+5)}/4q^l}$ . Even for a single sequence  $\{G_n\}$ , the problem of finding a complete solution of (3) may be far from trivial. For instance, it had been known for a while that the only squares and cubes in the Fibonacci sequence are  $\{F_0 = 0, F_1 = 1, F_2 = 1, F_{12} = 144\}$  and  $\{F_0 = 0, F_1 = 1, F_2 = 1, F_6 = 8\}$  respectively, but to prove that those are the only powers, Bugeaud, Mignotte and Siksek [2] had to combine the classical approach with modular methods similar to those used by Wiles to prove Fermat's last theorem.

Let us fix the exponent  $q$ . We present an elementary procedure, introduced in [1], to approximate the solutions of (3) in the following sense. The procedure outputs a large integer  $N = N_q$  and a relatively small set  $\mathcal{J} \subset \mathbb{Z}/N\mathbb{Z}$  such that if  $G_n$  solves (3) then  $\bar{n} = n \bmod N \in \mathcal{J}$ . The actual computations show that the procedure "converges" rather quickly and in many cases yields  $\mathcal{J} = \emptyset$  showing the absence of solutions for the corresponding equation.

The procedure is explained in section 2 followed by some heuristics in section 3. A final section gives a few example of actual computations. We test all non-trivial sequences  $\{G_n\}$  with positive parameters  $A$  and  $B$ , and non-negative initial values  $G_0$  and  $G_1$  with  $A + B \leq 4$  and  $\max\{G_0, G_1\} \leq 9$  up to shift-equivalence (see Definition 2.1). There are two kinds of tables. Tables 1 to 6 show the result of running the procedure in search of  $q$ -powers, for

$q \in \{3, 5, 7, 11, 13, 17\}$ . Tables 7 to 12 list the values of  $k$  for which (3) with  $q = 3$  or  $q = 5$  has no solutions for  $2 \leq k \leq 30$  and  $q$ -power free. In particular, the following result remains proved.

**Theorem 1.3.** *Let  $\{G_n\}$  be a binary recurrence sequence. The equation  $G_n = kx^q$  has no solutions in all cases labelled  $\emptyset$  in Tables 1 to 6 and for all values  $(q, k)$  listed in Tables 7 to 12 below.*

An analysis of the tables 1–6 shows that in many cases, up to replacing  $N$  by a large divisor, the set  $\mathcal{J}$  consists of just one element, so that up to shift-equivalence we may assume that  $\mathcal{J} = \{\bar{0}\}$ . The following question arises naturally. Suppose that there is a (large) integer  $N$  such that a solution of  $G_n = kx^q$  can occur only for  $n \equiv 0 \pmod{N}$ . Can we obtain further information on the set of solutions from arithmetic properties of the triple  $(k, q, N)$ ? In particular, can we deduce the finiteness of the number of solutions independently of Baker's theory?

## 2 The procedure

We shall assume that  $AB \neq 0$ . The binary recurrence sequence (1) extends uniquely to a function  $\mathbb{Z} \rightarrow \mathbb{Z}[1/B]$  in such a way that the recurrence relation  $G_{n+2} = AG_{n+1} + BG_n$  remains valid for all  $n \in \mathbb{Z}$ . Namely, set inductively

$$G_{-n} = -\frac{A}{B}G_{-n+1} + \frac{1}{B}G_{-n+2} \quad \text{for all } n > 0.$$

**Definition 2.1.** *Two extended binary recurrence sequences  $\{G_n\}$  and  $\{G'_n\}$  are called shift-equivalent if there exists  $k \in \mathbb{Z}$  such that  $G'_n = G_{n+k}$  for all  $n \in \mathbb{Z}$ .*

**Proposition 2.2.** *1. Two sequences not of the form  $\{g\mu^n\}$  are shift-equivalent if and only if they share four equal consecutive terms.*

*2. The sequences  $\{g\mu^n\}$  and  $\{G_n\}$  are shift-equivalent if and only if  $G_n = g'\mu^n$  with  $g' = g\mu^k$  for some  $k \in \mathbb{Z}$ .*

*Proof.* The sequences  $\{G_n\}_{n \in \mathbb{Z}}$  and  $\{G'_n\}_{n \in \mathbb{Z}}$  with same parameters  $A$  and  $B$  are shift-equivalent if and only if they have a common segment of length 2,  $G'_r = G_s$  and  $G'_{r+1} = G_{s+1}$  for some  $r, s \in \mathbb{Z}$ . When  $G_k^2 \neq AG_kG_{k-1} + BG_{k-1}^2$  for some (or, equivalently, all)  $k \in \mathbb{Z}$  the parameters  $A$  and  $B$  can be recovered from the consecutive terms  $G_{k-1}, \dots, G_{k+2}$  by solving the linear equations

$$\begin{cases} G_{k+2} &= AG_{k+1} + BG_k \\ G_{k+1} &= AG_k + BG_{k-1} \end{cases}$$

This proves part 1 once we observe that the sequences of the form  $\{g\mu^n\}$  are precisely those for which  $G_k^2 = AG_kG_{k-1} + BG_{k-1}^2$ . Part 2 is immediate.  $\square$

The previous fact remains true for  $R$ -valued sequences, where  $R$  is any domain of characteristic prime to  $B$ .

**Definition 2.3.** Let  $\ell$  be a prime number,  $(\ell, B) = 1$ . The reduction modulo  $\ell$  of the  $\mathbb{Z}$ -valued binary recurrence sequence (1) is the sequence  $\{\overline{G}_n\}$  where  $\overline{G}_n \in \mathbb{F}_\ell = \mathbb{Z}/\ell\mathbb{Z}$  is the class of  $G_n$ .

The reduced sequence  $\{\overline{G}_n\}$  is an  $\mathbb{F}_\ell$ -valued binary recurrence sequence with parameters  $\overline{A}$  and  $\overline{B} \neq 0$  and initial values  $\overline{u}, \overline{v}$ . Its extension  $\{\overline{G}_n\}_{n \in \mathbb{Z}}$  is the reduction modulo  $\ell$  of the extension  $\{G_n\}$ . The following very simple fact is the basis of the procedure.

**Proposition 2.4.** Let  $\{\overline{G}_n\}$  be an extended  $\mathbb{F}_\ell$ -valued binary recurrence sequence. Then  $\{\overline{G}_n\}$  is periodic.

*Proof.* Since there are only a finite number of pairs  $(a, b) \in \mathbb{F}_\ell \times \mathbb{F}_\ell$ , there must be integers  $r \neq s$  such that  $\overline{G}_r = \overline{G}_s$  and  $\overline{G}_{r+1} = \overline{G}_{s+1}$ . If  $0 \neq k = s - r$ , an obvious induction shows that the sequences  $\{\overline{G}_n\}$  and  $\{\overline{G}_{n+k}\}$  coincide.  $\square$

**Definition 2.5.** For a prime number  $\ell$ , let  $\pi_\ell$  be the minimal period of the extended  $\mathbb{F}_\ell$ -valued reduced sequence  $\{\overline{G}_n\}$ , i.e.

$$\pi_\ell = \min \{k \in \mathbb{Z}^{>0} \text{ such that } \overline{G}_{n+k} = \overline{G}_n \text{ for all } n \in \mathbb{Z}\}.$$

**Proposition 2.6.** Let  $\ell$  be a prime number. The period  $\pi_\ell$  is a divisor of

1.  $\ell(\ell - 1)$ , if  $\Delta = 0$  or if  $\Delta$  is not a square in  $\mathbb{Z}$  with  $\ell \mid \Delta$ ;
2.  $\ell - 1$ , if  $\Delta$  is a non-zero square or if  $(\frac{\Delta}{\ell}) = 1$ ;
3.  $\ell^2 - 1$ , if  $\Delta$  is not a square and  $(\frac{\Delta}{\ell}) = -1$

*Proof.* From the description (2), the period  $\pi_\ell$  is the order of the cyclic quotient group  $\langle \overline{M} \rangle / \langle \overline{M} \rangle \cap S_{\overline{u}, \overline{v}}$  where  $\overline{M} \in \text{GL}_2(\mathbb{F}_\ell)$  is the reduction modulo  $\ell$  of  $M = \begin{pmatrix} A & B \\ 1 & 0 \end{pmatrix}$  and  $S_{\overline{u}, \overline{v}}$  is the stabilizer of the vector  $(\frac{\overline{u}}{\overline{v}})$  under the tautological action of  $\text{GL}_2(\mathbb{F}_\ell)$  on  $(\mathbb{F}_\ell)^2$ . Thus  $\pi_\ell \mid \text{ord}(\overline{M})$ .

If  $\Delta = 0$  then  $K = \mathbb{Q}$ ,  $\lambda_1 = \lambda_2 = \lambda \in \mathbb{Z}$  and  $M \sim \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ , whose order modulo  $\ell$  is  $\ell(\ell - 1)$ .

If  $\Delta \neq 0$  the eigenvalues are different, so  $M \sim \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$  with  $\lambda_1, \lambda_2 \in \mathbb{Q}$  if  $\Delta$  is a square or  $\lambda_2 = \tau(\lambda_1)$  otherwise. Hence  $\text{ord}(\overline{M})$  is the least common divisors of the orders of  $\overline{\lambda}_1$  and  $\overline{\lambda}_2$  as elements of  $(\mathcal{O}_K/\ell\mathcal{O}_K)^\times$ . Thus the other cases follow recalling that

$$(\mathcal{O}_K/\ell\mathcal{O}_K)^\times \simeq \begin{cases} \mathbb{F}_\ell^\times & \text{if } K = \mathbb{Q}, \\ \mathbb{F}_\ell^\times \times \mathbb{F}_\ell^\times & \text{if } K \text{ quadratic and } \ell \text{ split}, \\ \mathbb{F}_{\ell^2}^\times & \text{if } K \text{ quadratic and } \ell \text{ inert}, \\ (\mathbb{F}_\ell[X]/(X^2))^\times & \text{if } K \text{ quadratic and } \ell \text{ ramified}. \end{cases}$$

$\square$

The procedure goes as follows.

**Step 1:** Input the defining data  $(u, v, A, B)$ , the equation data  $(k, q)$  and fix a cutoff value  $C_{\text{off}} > 0$ .

**Step 2:** Consider the primes  $\ell_1 < \dots < \ell_r \leq C_{\text{off}}$  satisfying the following three conditions:

1.  $\ell_i$  does not divide  $Bk$  for all  $i = 1, \dots, r$ ;
2.  $\ell_i \equiv 1 \pmod{q}$  for all  $i = 1, \dots, r$ ;
3. if we set  $n_1 = \pi_{\ell_1}$  and define  $n_{i+1}$  for  $i = 1, \dots, r-1$  inductively as  $n_{i+1} = \text{lcm}(n_i, \pi_{\ell_{i+1}})$ , then  $n_{i+1}/n_i < q$  for all  $i = 1, 2, \dots, r-1$ .

**Step 3:** Construct inductively sets  $\mathcal{J}_i \subset \mathbb{Z}/n_i\mathbb{Z}$  as follows:

1.  $\mathcal{J}_1 = \{\overline{n} \in \mathbb{Z}/n_1\mathbb{Z} \text{ such that } \overline{G_n}/\overline{k} \in (\mathbb{F}_{\ell_1})^q\}$ ;
2. for  $i = 1, 2, \dots, r-1$ , given  $\mathcal{J}_i$  first set

$$\mathcal{J}_{i+1}^\# = \{\overline{n} \in \mathbb{Z}/n_{i+1}\mathbb{Z} \text{ such that } n \bmod n_i \in \mathcal{J}_i\}$$

and then let

$$\mathcal{J}_{i+1} = \mathcal{J}_{i+1}^\# - \{\overline{n} \text{ such that } \overline{G_n}/\overline{k} \notin (\mathbb{F}_{\ell_{i+1}})^q\}$$

**Step 4:** If  $\mathcal{J}_{r'} = \emptyset$  for some  $r' \leq r$  the procedure stops, else let  $N = n_r$  and output  $\mathcal{J} = \mathcal{J}_r \subset \mathbb{Z}/N\mathbb{Z}$ .

The reason for the conditions on the primes  $\ell_i$  is the following. The subgroup  $(\mathbb{F}_\ell^\times)^q$  of  $q$ -powers in the multiplicative group  $\mathbb{F}_\ell^\times$  is proper if and only if  $q \mid \ell-1$ , and in this case consists of  $(\ell-1)/q$  elements. Thus, the number of  $q$ -powers in  $\mathbb{F}_\ell$  is  $(q + \ell - 1)/q$  and on average we can expect that at each step

$$|\mathcal{J}_{i+1}| \cong \frac{q + \ell_{i+1} - 1}{q\ell_{i+1}} |\mathcal{J}_{i+1}^\#|.$$

Since  $|\mathcal{J}_{i+1}^\#| = (n_{i+1}/n_i)|\mathcal{J}_i|$ , by forcing  $n_{i+1}/n_i \leq q-1$  and observing that  $\lim_{i \rightarrow \infty} \frac{q + \ell_i - 1}{q\ell_i}(q-1) < 1$  we can expect that eventually  $|\mathcal{J}_{i+1}| < |\mathcal{J}_i|$  on average, so that the procedure should eventually produce an empty set of indices when the equation (3) has no solutions.

**Remark 2.7.** *The necessity of imposing condition 3 in Step 2 makes the procedure unsuited for the case  $q = 2$ .*

### 3 Heuristic density estimates

The support of  $n \in \mathbb{Z}$  is the set  $\text{Supp}(n) = \{p \text{ prime such that } p \mid n\}$ . Fix an integer  $m \geq 2$  and let  $\mathcal{P}_m = \{\ell \text{ prime such that } \max(\text{Supp}(\pi_\ell)) \leq m\}$  and

$$\mathcal{P}'_m = \{\ell \text{ prime such that } \max(\text{Supp}(\text{ord}_\ell(\overline{M}))) \leq m\}.$$

Also, let  $\mathcal{P}_{m,q} = \{\ell \in \mathcal{P}_m \text{ such that } \ell \equiv 1 \pmod{q}\}$  and

$$\mathcal{P}'_{m,q} = \{\ell \in \mathcal{P}'_m \text{ such that } \ell \equiv 1 \pmod{q}\}.$$

The sets  $\mathcal{P}'_m$  and  $\mathcal{P}'_{m,q}$  depend on the coefficients  $A$  and  $B$ , while the sets  $\mathcal{P}_m$  and  $\mathcal{P}_{m,q}$  depend also on the vector  $\vec{v} = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{Z}^2$  of initial values. Since  $\pi_\ell \mid \text{ord}_\ell(\overline{M})$ , we have that  $\mathcal{P}'_m \subseteq \mathcal{P}_m$  and  $\mathcal{P}'_{m,q} \subseteq \mathcal{P}_{m,q}$ . The primes  $\ell_1, \ell_2, \dots$  of Step 2 are in  $\mathcal{P}_{q-1,q}$ . We shall show that in the case of a non-degenerate binary recurrence sequence with non-zero initial vector  $\vec{v}$ , a variation of the classical Artin heuristics, under the usual independence hypotheses, yields that the expected density of the sets  $\mathcal{P}_m$ , and hence  $\mathcal{P}_{m,q}$ , is 0.

Let assume first that  $K = \mathbb{Q}$  and, for the sake of uniformity of the argument, also that  $\min\{|\lambda_1|, |\lambda_2|\} \geq 2$ . Let  $\Sigma_0$  be the finite set of primes containing 2 and the primes dividing  $\lambda_1 \lambda_2$ . Consider a prime  $\ell \notin \Sigma_0$  and write  $\ell - 1 = ab$  where  $\max\{\text{Supp}(a)\} \leq m$  and  $\min\{\text{Supp}(b)\} > m$ . Then  $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathbb{F}_\ell^\times \times \mathbb{F}_\ell^\times$  and

$$\begin{aligned} \max(\text{Supp}(\text{ord}_\ell(\overline{M}))) \leq m &\iff \overline{\lambda}_1 \text{ and } \overline{\lambda}_2 \text{ are } b\text{-powers in } \mathbb{F}_\ell^\times \\ &\iff \overline{\lambda}_1 \text{ and } \overline{\lambda}_2 \text{ are } p^r\text{-powers in } \mathbb{F}_\ell^\times \text{ for} \\ &\quad \text{all primes } p > m \text{ such that } p^r \parallel \ell - 1 \end{aligned}$$

Since the primes  $\ell \equiv 1 \pmod{p^r}$  are precisely those that split completely in the cyclotomic extension  $\mathbb{Q} \subset \mathbb{Q}(\mu_{p^r})$ , we can rephrase the last condition in terms of the extensions in the diagram

$$\begin{array}{ccc} \mathbb{Q}(\mu_{p^{r+1}}) & & \mathbb{Q}(\mu_{p^r}, \sqrt[p^r]{\lambda_1}, \sqrt[p^r]{\lambda_2}) \\ & \searrow \quad \swarrow & \\ & \mathbb{Q}(\mu_{p^r}) & \\ & \downarrow & \\ & \mathbb{Q} & \end{array} \quad \begin{array}{l} \text{I} \quad \quad \quad \text{II} \end{array} \quad (4)$$

Namely,  $\overline{\lambda}_1$  and  $\overline{\lambda}_2$  are in  $(\mathbb{F}_\ell^\times)^{p^r}$  and  $p^r \parallel \ell - 1$  if and only if  $\ell \in \Sigma'_{p,r}$ , where  $\Sigma'_{p,r} = \{\text{primes } \ell \text{ that split completely in II and do not split completely in I}\}$ . By construction,  $\Sigma'_{p,r} \cap \Sigma'_{p,r'} = \emptyset$  if  $r \neq r'$ , and if we let  $\Sigma'_p = \cup_{r \geq 1} \Sigma'_{p,r}$  then

$$\mathcal{P}'_m = \bigcap_{p > m} \Sigma'_p. \quad (5)$$

The following proposition is a straightforward application of Kummer's theory to the situation of diagram (4).

**Proposition 3.1.** *Suppose  $p \notin \Sigma_0$ . Then:*

1.  $[\mathbb{Q}(\mu_{p^r}, \sqrt[p^r]{\lambda_i}) : \mathbb{Q}(\mu_{p^r})] = p^r$  for  $i = 1, 2$ ;
2.  $\mathbb{Q}(\mu_{p^r}, \sqrt[p^r]{\lambda_1}) \cap \mathbb{Q}(\mu_{p^r}, \sqrt[p^r]{\lambda_2}) = \mathbb{Q}(\mu_{p^r})$ ;

3.  $\text{Gal}(\mathbb{Q}(\mu_{p^r}, \sqrt[p^r]{\lambda_1}, \sqrt[p^r]{\lambda_2})/\mathbb{Q}(\mu_{p^r})) \simeq (\mathbb{Z}/p^r\mathbb{Z})^2$ ;
4.  $\mathbb{Q}(\mu_{p^{r+1}}) \cap \mathbb{Q}(\mu_{p^r}, \sqrt[p^r]{\lambda_1}, \sqrt[p^r]{\lambda_2}) = \mathbb{Q}(\mu_{p^r})$ .

In particular, for  $p \notin \Sigma_0$  point 4 says that  $\Sigma'_{p,r} \neq \emptyset$  and by Čebotarev's theorem the expected density of  $\Sigma'_{p,r}$  is

$$\begin{aligned} \delta(\Sigma'_{p,r}) &= \left(1 - \frac{1}{[\mathbb{Q}(\mu_{p^{r+1}}) : \mathbb{Q}(\mu_{p^r})]}\right) \frac{1}{[\mathbb{Q}(\mu_{p^r}, \sqrt[p^r]{\lambda_1}, \sqrt[p^r]{\lambda_2}) : \mathbb{Q}]} \\ &= \frac{p-1}{p} \frac{1}{p^{3r-1}(p-1)} = \frac{1}{p^{3r}} \end{aligned}$$

so that  $\delta(\Sigma'_p) = \sum_{r \geq 1} p^{-3r} = 1/(p^3 - 1)$ . Applying the independence assumption to (5) yields the expected value

$$\delta(\mathcal{P}'_m) = \prod_{\substack{p > m \\ p \in \Sigma_0}} \delta(\Sigma'_p) \prod_{\substack{p > m \\ p \notin \Sigma_0}} \frac{1}{p^3 - 1} = 0.$$

Let  $\ell \in \mathcal{P}_m - \mathcal{P}'_m$ ,  $\ell \notin \Sigma_0$ . Then  $M^{\pi_\ell} \not\equiv I \pmod{\ell}$  and yet

$$M^{\pi_\ell} \vec{v} \equiv \vec{v} \pmod{\ell} \tag{6}$$

In order for this to be possible, the matrix  $M^{\pi_\ell} \pmod{\ell}$  must admit 1 as an eigenvalue. Thus a prime  $\ell \notin \Sigma_0$  is in  $\mathcal{P}_m - \mathcal{P}'_m$  if and only if the following two conditions are satisfied.

- C1. Exactly one of the eigenvalues  $\lambda_1, \lambda_2$  is a  $b$ -power in  $\mathbb{F}_\ell^\times$ . Equivalently, exactly one of the eigenvalues  $\lambda_1, \lambda_2$  is a  $p^r$ -power in  $\mathbb{F}_\ell^\times$  for all  $p^r \parallel \ell - 1$  with  $p > m$ .
- C2. If  $\lambda$  is the eigenvalue of condition C1, then  $\vec{v} \pmod{\ell} \in E_\lambda$  where  $E_\lambda \subset (\mathbb{Z}/\ell\mathbb{Z})^2$  is the  $\lambda$ -eigenspace of  $M \pmod{\ell}$ .

Denote  $\mathcal{P}_m^b$  the set of primes satisfying condition C1 only. As above  $\mathcal{P}_m^b = \bigcap_{p > m} \Sigma_p$  where  $\Sigma_p = \bigcup_{r \geq 1} \Sigma_{p,r}$  is a disjoint union with

$$\Sigma_{p,r} = \left\{ \ell \text{ that split completely in one extension } \mathbb{Q} \subset \mathbb{Q}(\mu_{p^r}, \sqrt[p^r]{\lambda_j}), \right. \\ \left. j = 1, 2, \text{ but not in both or in I of diagram 4} \right\}.$$

Given  $T > 0$ , let  $\binom{0}{0} \notin \mathcal{V}_T \subset \mathbb{Z}^2$  be a finite set such that the restriction of the product of quotient maps

$$\mathcal{V}_T \longrightarrow \prod_{\substack{\ell \leq T \\ \ell \in \mathcal{P}_m^b}} (\mathbb{Z}/\ell\mathbb{Z})^2$$

is a bijection and  $\mathcal{V}_T \subseteq \mathcal{V}_{T'}$  for  $T \leq T'$ . Then, denoting (as usual)  $\pi(T)$  the number of primes less than  $T$  and making explicit the dependence of  $\mathcal{P}_m$  on the

initial vector,

$$\delta_T := \frac{1}{|\mathcal{V}_T|} \sum_{\vec{v} \in \mathcal{V}_T} \frac{|\{\ell \in \mathcal{P}(\vec{v})_m \text{ such that } \ell \leq T\}|}{\pi(T)} = \frac{1}{\pi(T)} \sum_{\substack{\ell \leq T \\ \ell \in \mathcal{P}_m^b}} \frac{1}{\ell}$$

because  $|E_\lambda| = \ell$ . Thus,  $\delta = \lim_{T \rightarrow \infty} \delta_T$  is the average density of the sets  $\mathcal{P}(\vec{v})_m$  for  $\vec{v} \in \bigcup_T \mathcal{V}_T$ . On the other hand,  $\delta_T < \pi(T)^{-1} \sum_{n=1}^T 1/n$  and the well-known asymptotics  $\pi(T) \sim T \log(T)^{-1}$  and  $\sum_{n=1}^T 1/n \sim \log(T)$  yield  $\delta = 0$ . Since the set  $\mathcal{V}_T$  can be constructed so to contain any given  $0 \neq \vec{v} \in \mathbb{Z}^2$ , we get an estimated density

$$\delta(\mathcal{P}(\vec{v})_m) = 0, \text{ for all } \vec{v} \neq 0, \quad \text{if } K = \mathbb{Q}.$$

Let us assume now that  $K$  is quadratic and let  $\lambda = \lambda_1$ . Note that non-degeneracy is equivalent to the subgroup  $\langle \lambda, \lambda^\tau \rangle < K^\times$  being free of rank 2. This time let  $\Sigma_0$  be the finite set of primes containing 2, the primes dividing  $N_{K/\mathbb{Q}}(\lambda)$ , the primes such that  $K \subset \mathbb{Q}(\mu_{\ell^\infty})$  and the primes that are ramified in  $K$ . Let  $\ell \notin \Sigma_0$ . If  $\ell$  is split in  $K$ , then  $\bar{\lambda} \in (\mathcal{O}_K/\ell\mathcal{O}_K)^\times \simeq \mathbb{F}_\ell^\times \times \mathbb{F}_\ell^\times$ . The situation is very similar to the case  $K = \mathbb{Q}$  and we omit the details.

If  $\ell$  is inert in  $K$ , then  $\bar{\lambda} \in (\mathcal{O}_K/\ell\mathcal{O}_K)^\times \simeq \mathbb{F}_{\ell^2}^\times$ . Write  $\ell^2 - 1 = ab$  where  $\max\{\text{Supp}(a)\} \leq m$  and  $\min\{\text{Supp}(b)\} > m$ . Then

$$\begin{aligned} \max(\text{Supp}(\text{ord}_\ell(\bar{M}))) \leq m &\iff \bar{\lambda} \text{ is a } b\text{-power in } \mathbb{F}_{\ell^2}^\times \\ &\iff \bar{\lambda} \text{ is a } p^r\text{-powers in } \mathbb{F}_{\ell^2}^\times \text{ for all} \\ &\quad \text{primes } p > m \text{ such that } p^r \parallel \ell^2 - 1 \end{aligned}$$

Let  $\Sigma'_{p,r}$  be the set of primes satisfying the latter condition at  $p$ . Consider the diagram of Galois extensions

$$\begin{array}{ccccc} & & & & K(\mu_{p^r}, \sqrt[p^r]{\lambda}, \sqrt[p^r]{\lambda^\tau}) \\ & & & \nearrow \Gamma'_K & \\ K(\mu_{p^r}) & & & \Gamma_K & \\ & \searrow H & & \text{III} & \\ \mathbb{Q}(\mu_{p^r}) & & K & & \\ & \searrow & & \Gamma & \\ & & & & \mathbb{Q} \end{array} \quad (7)$$

Then

$$\Sigma'_{p,r} = \{\text{primes } \ell \text{ that split completely in III and such that } \ell \not\equiv \pm 1 \pmod{p^{r+1}}\}.$$

Again,  $\Sigma'_{p,r} \cap \Sigma'_{p,r'} = \emptyset$  if  $r \neq r'$  and if we let  $\Sigma'_p = \bigcup_{r \geq 1} \Sigma'_{p,r}$ , then

$$\tilde{\mathcal{P}}'_m = \{\ell \in \mathcal{P}'_m \text{ such that } \ell \text{ is inert in } K\} = \bigcap_{p > m} \Sigma'_p. \quad (8)$$



The analogous of proposition 3.1 is the following

**Proposition 3.2.** *Suppose  $p \notin \Sigma_0$  and  $\lambda/\lambda^\tau$  not a root of 1. Then:*

1.  $[K(\mu_{p^r}, \sqrt[p^r]{\lambda}) : K(\mu_{p^r})] = [K(\mu_{p^r}, \sqrt[p^r]{\lambda^\tau}) : K(\mu_{p^r})] = p^r$ ;
2.  $K(\mu_{p^r}, \sqrt[p^r]{\lambda}) \cap K(\mu_{p^r}, \sqrt[p^r]{\lambda^\tau}) = K(\mu_{p^r})$ ;
3.  $\text{Gal}(K(\mu_{p^r}, \sqrt[p^r]{\lambda}, \sqrt[p^r]{\tau(\lambda)})/K(\mu_{p^r})) \simeq (\mathbb{Z}/p^r\mathbb{Z})^2$ ;
4.  $\mathbb{Q}(\mu_{p^{r+1}}) \cap K(\mu_{p^r}, \sqrt[p^r]{\lambda}, \sqrt[p^r]{\lambda^\tau}) = \mathbb{Q}(\mu_{p^r})$ .

To estimate the density of the primes in  $\Sigma'_{p,r}$ , observe that an inert prime  $\ell$  splits completely in the extension (I) of diagram (7) if and only if a Frobenius element  $\sigma \in \text{Frob}_{K(\mu_{p^r}, \sqrt[p^r]{\lambda}, \sqrt[p^r]{\lambda^\tau})/K}(\ell) \subset \Gamma$  satisfies the following conditions:

$$\sigma^2 = \text{id} \quad \text{and} \quad \sigma|_K = \tau.$$

These conditions define a conjugacy class  $C \subset \Gamma$  and by Čebotarev's theorem we need to estimate its size. The exact sequences of Galois groups

$$1 \longrightarrow \Gamma_K \longrightarrow \Gamma \longrightarrow \langle \tau \rangle \longrightarrow 1$$

and

$$1 \longrightarrow \Gamma'_K \longrightarrow \Gamma_K \longrightarrow H \longrightarrow 1$$

split, so that  $\Gamma \simeq \Gamma_K \rtimes \langle \tau \rangle \simeq (\Gamma'_K \rtimes H) \rtimes \langle \tau \rangle$ . The extension  $\mathbb{Q} \subset K(\mu_{p^r})$  is abelian with Galois group isomorphic to  $G = H \rtimes \langle \tau \rangle$  so that we get  $\Gamma \simeq \Gamma'_K \rtimes G$ . Since  $H$  is cyclic (of even order  $p^{r-1}(p-1)$ ) there are 2 elements of order 2 in  $G$  restricting to  $\tau$  and finally

$$|C| \leq 2|\Gamma'_K| = 2p^{2r}.$$

Combining this estimate with Dirichlet's theorem of primes in arithmetic progression under the independence assumptions we get

$$\delta(\Sigma'_{p,r}) \leq \left( \frac{p-1}{p} \right) \frac{2p^{2r}}{2p^{3r-1}(p-1)} = \frac{1}{p^r}.$$

Thus,  $\delta(\Sigma'_p) \leq \sum_{r \geq 1} p^{-r} = 1/(p-1)$  and finally, from (8) and recalling that the inert primes have density 1/2,

$$\delta(\tilde{\mathcal{P}}'_m) = \frac{1}{2} \prod_{\substack{p > m \\ p \in \Sigma_0}} \delta(\Sigma'_p) \prod_{\substack{p > m \\ p \notin \Sigma_0}} \frac{1}{p-1} = 0.$$

The analysis of the set  $\mathcal{P}_m - \mathcal{P}'_m$  follows the same lines of the  $K = \mathbb{Q}$  situation in the case of a split prime  $\ell$  and we, again, omit the details. When  $\ell$  is inert the basically trivial observation that  $\lambda$  is a  $b$ -power if and only if  $\bar{\lambda}$  is a  $b$ -power implies at once that

$$\pi_\ell = \text{ord}_\ell(\bar{M}) \quad \text{if } \ell \text{ is inert.}$$

In other words, the set  $\mathcal{P}_m - \mathcal{P}'_m$  consists only of split primes or primes in  $\Sigma_0$  and the heuristic estimate

$$\delta(\mathcal{P}(\vec{v})_m) = 0, \text{ for all } \vec{v} \neq 0, \quad \text{if } K \text{ is quadratic,}$$

follows.

## 4 Tables

We implemented the procedure using the Maple 12 package and let it run on a MacBook. The tables in this section report some of these computations, done with a cutoff value  $C_{\text{off}} = 10000$ .

We consider all sequences up to shift-equivalence with positive parameters  $A$  and  $B$  such that  $A + B \leq 4$  and non-negative initial values  $G_0$  and  $G_1$  such that  $\max\{G_0, G_1\} \leq 9$ .

Tables 1–6 give the results of applying the procedure in search of pure powers for prime exponents  $q$  with  $3 \leq q \leq 17$ . Each table shows at the beginning the values  $N_q$  which depend only on  $A$ ,  $B$  and the cutoff value. The tables contain 4 types of entries:

1. an entry  $\emptyset$  indicates that the procedure outputs the empty set, i.e. that the corresponding sequence does not contain  $q$ -th powers;
2. an entry  $\{a\}$  indicates that the procedure shows that the only  $q$ -th powers in the corresponding sequence  $\{G_n\}$  can occur only for  $n \equiv a \pmod{N_q}$ ;
3. an entry  $\{a\}_m$  indicates that the procedure shows that the only  $q$ -th powers in the corresponding sequence  $\{G_n\}$  can occur only for  $n \equiv a \pmod{N_q/m}$ ;
4. an entry  $m$  indicates that the procedure final output was a set of  $m$  different possible classes modulo  $N_q$  for indices  $n$  with  $G_n$  a  $q$ -th power, not coming from the same class modulo  $N_q/m$ .

Tables 7–12 list the  $q$ -power free values  $2 \leq k \leq 30$  for which the procedure shows that the equation (3) has no solutions.

TABLE 1  
 $q$ -powers in sequences with  $A = 1$  and  $B = 1$

$$\begin{aligned} N_3 &= 186624, N_5 = 15552000, N_7 = 127008000, \\ N_{11} &= 3841992000, N_{13} = 43286443200 \\ N_{17} &= 68235175008000 \end{aligned}$$

$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
0	1	96	42	18	14	20	26
<i>continued on next page</i>							

Table 1: $q$ -powers in sequences with $A = 1, B = 1$ (continued from previous page)							
$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q=17$
0	2	24	6	$\{0\}_2$	6	18	6
0	3	$\{0\}_6$	$\{0\}_4$	6	$\{0\}_4$	$\{0\}_2$	$\{0\}_2$
0	4	24	14	$\{0\}_2$	$\{0\}_6$	6	18
0	5	$\{0\}_4$	$\{0\}_4$	$\{0\}_4$	6	$\{0\}_2$	18
0	6	$\{0\}_6$	$\{0\}_2$	12	$\{0\}_2$	$\{0\}_2$	$\{0\}_2$
0	7	18	6	$\{0\}_4$	6	$\{0\}_2$	$\{0\}_2$
0	8	96	$\{0\}_4$	$\{0\}_2$	$\{0\}_2$	$\{0\}_2$	10
0	9	22	6	6	6	$\{0\}_2$	$\{0\}_2$
1	3	48	8	8	16	8	32
1	4	62	4	4	$\{0\}_2$	4	4
1	5	30	4	$\{0\}_2$	4	4	12
1	6	4	$\{0\}_2$	$\{0\}_4$	8	4	6
1	7	20	$\{0\}_2$	$\{0\}_2$	48	$\{0\}_2$	12
1	8	64	4	$\{0\}_2$	$\{0\}_2$	4	32
1	9	32	$\{0\}_2$	$\{0\}_2$	8	4	48
2	5	62	4	4	$\{-2\}_2$	4	4
2	6	24	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	7	$\{-3\}_2$	$\emptyset$	$\{-9\}_2$	$\emptyset$	$\emptyset$	$\emptyset$
2	8	$\{1\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	7	30	4	$\{-2\}_2$	4	4	12
3	8	$\{1\}_2$	$\emptyset$	$\{7\}_2$	$\emptyset$	$\emptyset$	$\emptyset$
3	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	9	4	$\{-2\}_2$	$\{-2\}_4$	8	4	6
6	4	$\{-2\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	5	20	$\{-1\}_2$	$\{-1\}_2$	48	$\{-1\}_2$	12
7	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	2	68	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	3	52	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	4	40	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	5	52	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	6	68	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	7	32	$\{-1\}_2$	$\{-1\}_2$	8	4	48
9	1	44	16	$\{1\}_2$	18	4	4
9	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	3	16	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
continued on next page							

Table 1: $q$ -powers in sequences with $A = 1, B = 1$ (continued from previous page)							
$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
9	6	16	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	8	44	16	$\{-1\}_2$	18	4	4

TABLE 2  
 $q$ -powers in sequences with  $A = 1$  and  $B = 2$

$$N_3 = 31104, N_5 = 7776000, N_7 = 111132000,$$

$$N_{11} = 295833384000, N_{13} = 86572886400$$

$$N_{17} = 393664471200$$

$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$p = 17$
0	1	4	8	5	15	6	3
0	2	2	2	4	17	4	3
0	3	$\{0\}$	$\{0\}$	$\{0\}$	15	$\{0\}$	$\{0\}$
0	4	2	6	4	5	13	2
0	5	$\{0\}_2$	$\{0\}$	3	3	$\{0\}$	$\{0\}$
0	6	3	5	$\{0\}$	3	$\{0\}$	9
0	7	$\{0\}_2$	$\{0\}$	$\{0\}$	3	$\{0\}$	3
0	8	3	$\{0\}$	3	3	3	3
0	9	5	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	4	4	$\{0\}$	2	2	3	$\{0\}$
1	5	$\{0\}$	$\{0\}$	3	8	$\{0\}$	4
1	6	2	3	2	3	6	8
1	7	5	3	6	4	6	4
1	8	6	3	$\{0\}$	4	2	6
1	9	$\{0\}$	$\{0\}$	4	4	$\{0\}$	4
2	3	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	8	$\{1\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	8	$\{1\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	2	$\{-1\}$	$\{-1\}$	3	8	$\{-1\}$	4
4	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

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Table 2: $q$ -powers in sequences with $A = 1, B = 2$ (continued from previous page)							
$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
4	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	7	$\{-3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	0	$\{1\}$	$\{1\}$	3	9	$\{1\}$	$\{1\}$
5	1	$\{1\}_2$	$\{1\}$	4	2	2	$\{1\}$
5	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	3	$\{-1\}$	3	2	3	6	8
5	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	8	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	0	$\{1\}_2$	3	3	3	$\{1\}_3$	$\{1\}_2$
6	1	2	$\{1\}$	$\{1\}_3$	$\{1\}$	$\{1\}$	6
6	2	$\{-6\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	4	2	2	4	2	3	3
6	5	$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	0	$\{1\}$	$\{1\}_2$	3	$\{1\}$	$\{1\}$	$\{1\}$
7	1	$\{1\}_2$	$\{1\}$	$\{1\}$	2	6	2
7	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	5	3	3	$\{-1\}$	4	2	6
7	6	$\emptyset$	$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	8	$\{1\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	0	2	$\{1\}$	9	27	3	$\{1\}$
8	1	4	$\{1\}$	$\{1\}$	2	2	$\{1\}$
8	2	$\{0\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	3	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	4	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	5	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	6	2	$\{-1\}$	4	4	$\{-1\}$	4
8	7	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	0	$\{1\}_2$	$\{1\}_2$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
9	1	$\{1\}$	2	2	$\{1\}$	2	2
9	2	$\{4\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	3	$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
continued on next page							

Table 2: $q$ -powers in sequences with $A = 1, B = 2$ (continued from previous page)							
$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
9	6	$\{-3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	7	$\{-1\}$	2	2	6	2	2
9	8	$\{1\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

TABLE 3  
 $q$ -powers in sequences with  $A = 2$  and  $B = 1$

$$N_3 = 41472, N_5 = 15552000, N_7 = 74088000$$

$$N_{11} = 12074832000, N_{13} = 519437318400$$

$$N_{17} = 787328942400$$

$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
0	1	6	26	26	14	38	14
0	2	$\{0\}_2$	18	$\{0\}_2$	6	6	6
0	3	$\{0\}_6$	$\{0\}_4$	10	6	6	10
0	4	12	6	$\{0\}_2$	6	6	6
0	5	$\{0\}_2$	12	6	$\{0\}_2$	$\{0\}_2$	18
0	6	6	$\{0\}_2$	$\{0\}_2$	$\{0\}_2$	$\{0\}_2$	6
0	7	$\{0\}_2$	6	$\{0\}_2$	18	$\{0\}_2$	$\{0\}_2$
0	8	6	24	$\{0\}_6$	6	6	6
0	9	$\{0\}_2$	6	10	6	$\{0\}_2$	$\{0\}_2$
1	1	8	44	16	8	84	8
1	2	6	26	26	14	38	14
1	3	8	44	16	8	84	8
1	4	8	6	6	6	$\{0\}_2$	4
1	5	12	$\{0\}_2$	4	4	4	$\{0\}_2$
1	6	$\{0\}_2$	10	6	12	6	4
1	7	4	8	$\{0\}_2$	4	12	8
1	8	6	4	12	8	4	8
1	9	$\{0\}_2$	8	$\{0\}_2$	6	4	$\{0\}_2$
2	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	3	8	6	6	6	$\{-1\}_2$	4
2	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	8	$\{1\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	9	$\{-2\}_8$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

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Table 3: $p$ -powers in sequences with $A = 2, B = 1$ (continued from previous page)							
$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
3	5	12	$\{-1\}_2$	7	4	4	$\{-1\}_2$
3	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	6	$\{-2\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	7	$\{-1\}_2$	10	6	12	6	4
5	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	8	$\{1\}_8$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	9	4	4	8	$\{-1\}_2$	4	8
6	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	3	8	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	4	$\{-1\}_4$	$\{3\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	8	$\{1\}_4$	$\{-3\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	9	8	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	2	$\{6\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	4	8	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	6	$\{-1\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	8	$\{1\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	1	12	4	$\{1\}_2$	6	$\{1\}_2$	$\{1\}_2$
8	2	10	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	3	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	4	16	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	5	$\{0\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	6	$\{0\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	7	$\{0\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	8	8	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	9	$\{0\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	1	6	$\{1\}_2$	16	4	$\{1\}_2$	12
continued on next page							

Table 3: $p$ -powers in sequences with $A = 2, B = 1$ (continued from previous page)							
$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
9	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	8	$\{1\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	9	8	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

TABLE 4  
 $q$ -powers in sequences with  $A = 1$  and  $B = 3$

$$N_3 = 46656, N_5 = 3888000, N_7 = 296335200$$

$$N_{11} = 658627200, N_{13} = 865728864000$$

$$N_{17} = 257297040000$$

$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
0	1	105	4	20	8	21	16
0	2	69	$\{0\}$	$\{0\}$	3	$\{0\}$	6
0	3	69	6	6	17	5	8
0	4	21	$\{0\}$	$\{0\}$	3	3	$\{0\}_2$
0	5	81	3	6	3	$\{0\}$	$\{0\}_2$
0	6	63	12	$\{0\}_2$	$\{0\}$	$\{0\}$	$\{0\}_2$
0	7	144	3	$\{0\}$	3	$\{0\}$	$\{0\}_2$
0	8	105	5	$\{0\}_6$	3	9	18
0	9	63	2	9	12	17	16
1	2	75	4	2	4	16	$\{0\}_2$
1	3	12	$\{0\}$	2	12	2	$\{0\}_4$
1	4	105	4	20	8	21	16
1	5	51	2	4	16	3	4
1	6	75	$\{0\}_2$	24	10	6	4
1	7	54	$\{0\}$	$\{0\}$	3	2	$\{0\}_2$
1	8	69	$\{0\}_2$	4	4	2	$\{0\}_2$
1	9	39	$\{0\}$	2	8	$\{0\}$	4
2	3	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	4	15	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	6	18	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

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Table 4: $q$ -powers in sequences with $A = 1$ , $B = 3$ (continued from previous page)							
$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
2	7	60	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	9	48	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	1	33	$\{1\}$	2	9	2	$\{1\}_4$
3	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	4	93	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	5	30	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	7	12	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	8	27	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	0	$\{1\}_3$	$\{1\}_4$	3	7	$\{1\}$	$\{1\}_2$
4	1	99	4	5	18	5	8
4	2	66	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	3	24	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	5	30	$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	6	36	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	8	30	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	9	72	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	0	42	$\{1\}_4$	3	$\{1\}$	$\{1\}$	$\{1\}_2$
5	1	9	2	2	6	12	8
5	2	87	7	28	26	24	10
5	3	12	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	4	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	6	39	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	7	63	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	9	96	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	0	108	$\{1\}$	$\{1\}$	$\{1\}$	3	$\{1\}_2$
6	1	33	$\{1\}$	8	10	6	$\{1\}_2$
6	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	3	33	$\{-1\}$	$\{-1\}$	3	2	$\{-1\}_2$
6	4	36	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	5	27	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	7	27	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	8	9	$\emptyset$	4	$\emptyset$	$\emptyset$	$\emptyset$
7	0	42	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	6
7	1	42	$\{1\}$	2	18	4	4
7	2	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	3	12	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	4	18	$\{-1\}_2$	4	4	2	$\{-1\}_2$
7	5	18	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	6	33	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	8	57	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
continued on next page							

Table 4: $q$ -powers in sequences with $A = 1$ , $B = 3$ (continued from previous page)							
$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
7	9	33	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	0	69	$\{1\}_4$	$\{1\}_3$	5	$\{1\}$	$\{1\}_2$
8	1	60	$\{1\}$	5	9	13	4
8	2	48	$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	3	54	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	4	177	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	5	48	$\{-1\}$	2	8	$\{-1\}$	4
8	6	36	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	7	45	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	9	24	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	0	105	$\{1\}$	5	9	$\{1\}$	6
9	1	36	6	$\{1\}$	2	2	8
9	2	36	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	4	36	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	5	12	$\{2\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	6	39	$\{-1\}$	2	9	2	$\{-1\}_4$
9	7	63	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	8	30	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

TABLE 5  
 $q$ -powers in sequences with  $A = 2$  and  $B = 2$

$$N_3 = 62208, N_5 = 7776000, N_7 = 177811200$$

$$N_{11} = 59166676800, N_{13} = 719566848000$$

$$N_{17} = 4374049680000$$

$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
0	1	32	6	16	18	214	24
0	2	28	33	24	60	300	36
0	3	$\{0\}_4$	$\{0\}$	$\{0\}_6$	18	54	10
0	4	24	5	$\{0\}_4$	54	54	50
0	5	$\{0\}_4$	$\{0\}$	$\{0\}_2$	18	54	6
0	6	$\{0\}_4$	$\{0\}_2$	6	18	94	6
0	7	$\{0\}_4$	5	$\{0\}_6$	86	22	6
0	8	32	$\{0\}$	22	18	230	$\{0\}_2$
0	9	$\{0\}_8$	5	6	6	26	6
1	1	32	8	26	60	198	56

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Table 5: $q$ -powers in sequences with $A = 2, B = 2$ (continued from previous page)							
$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
1	3	24	5	10	54	196	30
1	5	8	6	16	24	148	18
1	6	8	$\{0\}$	8	18	56	8
1	7	8	3	36	6	288	12
1	8	8	2	6	18	106	20
1	9	8	$\{0\}$	$\{0\}_2$	12	60	16
2	2	8	$\{-1\}$	$\{-1\}_2$	12	56	4
2	3	$\{-5\}_4$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	8	8	$\{-2\}$	$\{-2\}_2$	12	56	4
2	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	4	$\{-1\}_4$	4	8	12	72	12
3	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	7	$\{-2\}_4$	3	4	36	164	4
3	9	$\{-2\}_4$	3	$\{-2\}_2$	18	44	10
4	1	16	$\{1\}$	8	36	128	4
4	2	$\{5\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	3	8	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	5	$\emptyset$	$\emptyset$	8	$\emptyset$	$\emptyset$	$\emptyset$
4	6	8	$\{-1\}$	8	18	56	8
4	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	0	$\{1\}_4$	$\{1\}_2$	6	18	54	10
5	1	8	3	4	24	76	80
5	2	$\emptyset$	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	8	16	3	36	6	288	12
5	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	0	$\{1\}_4$	$\{1\}$	6	6	78	6
6	1	16	3	6	24	44	6
6	2	16	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
continued on next page							

Table 5: $q$ -powers in sequences with $A = 2, B = 2$ (continued from previous page)							
$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
6	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	5	8	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	8	$\{1\}_4$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	9	8	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	0	$\{1\}_4$	3	6	18	138	6
7	1	$\{1\}_4$	3	12	36	144	40
7	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	5	te-24	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	6	8	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	8	16	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	9	$\emptyset$	$\{2\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	0	28	5	$\{1\}_2$	54	26	18
8	1	20	2	6	12	58	8
8	2	8	$\emptyset$	8	$\emptyset$	$\emptyset$	$\emptyset$
8	3	$\{0\}_4$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	4	$\{0\}_4$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	5	$\{0\}_4$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	6	$\{0\}_4$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	7	12	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	8	32	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	9	12	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	0	$\{1\}_4$	3	12	54	46	10
9	1	$\{1\}_{12}$	10	12	12	80	64
9	2	$\{-1\}_4$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	7	$\emptyset$	$\{2\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	8	$\{1\}_4$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

TABLE 6  
 $q$ -powers in sequences with  $A = 3$  and  $B = 1$

$$N_3 = 93312, N_5 = 15552000, N_7 = 148176000$$

$$N_{11} = 46103904000, N_{13} = 432864432000$$

$$N_{17} = 102918816000$$

$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
0	1	70	10	6	6	10	6
0	2	22	$\{0\}_2$	$\{0\}_2$	$\{0\}_2$	$\{0\}_2$	6
0	3	22	$\{0\}_2$	30	6	$\{0\}_2$	$\{0\}_2$
0	4	44	$\{0\}_2$	$\{0\}_2$	36	$\{0\}_2$	$\{0\}_2$
0	5	22	$\{0\}_2$	36	$\{0\}_2$	$\{0\}_2$	$\{0\}_2$
0	6	44	$\{0\}_2$	12	$\{0\}_2$	$\{0\}_2$	$\{0\}_2$
0	7	22	$\{0\}_2$	6	12	$\{0\}_2$	$\{0\}_2$
0	8	70	$\{0\}_4$	$\{0\}_2$	6	6	6
0	9	180	$\{0\}_4$	10	6	$\{0\}_2$	6
1	1	44	6	6	92	10	6
1	2	44	6	6	92	10	6
1	5	24	4	16	24	32	4
1	6	62	$\{0\}_2$	16	8	6	$\{0\}_2$
1	7	80	$\{0\}_2$	48	6	12	4
1	8	114	4	$\{0\}_2$	6	4	$\{0\}_4$
1	9	4	$\{0\}_2$	16	18	4	$\{0\}_2$
2	2	18	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	4	18	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	5	24	4	16	24	32	4
2	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	8	62	$\{-1\}_2$	16	8	6	$\{-1\}_2$
4	2	$\emptyset$	$\{3\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	4	28	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

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Table 6: $q$ -powers in sequences with $A = 3, B = 1$ (continued from previous page)							
$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
4	8	28	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	4	24	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	6	$\emptyset$	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	7	14	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	8	14	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	9	$\emptyset$	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	7	14	$\emptyset$	$\{-3\}_2$	$\emptyset$	$\emptyset$	$\emptyset$
6	8	24	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	1	36	$\{1\}_2$	8	4	$\{1\}_2$	$\{1\}_2$
7	2	24	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	8	24	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	1	60	6	8	8	4	$\{1\}_2$
8	2	48	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	3	44	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	4	42	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	5	20	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	6	32	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	7	40	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	8	44	$\{2\}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	9	18	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	1	18	4	4	$\{1\}_2$	$\{1\}_2$	$\{1\}_2$
9	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	4	28	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
continued on next page							

Table 6: $q$ -powers in sequences with $A = 3, B = 1$ (continued from previous page)							
$G_0$	$G_1$	$q = 3$	$q = 5$	$q = 7$	$q = 11$	$q = 13$	$q = 17$
9	5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	6	24	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	8	32	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
9	9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

TABLE 7  
Values of  $q$ -power free constants  $2 \leq k \leq 30$  for which the equation  
 $G_n = kx^q$  has no solutions with  $A = 1, B = 1$ , and  $q = 3, 5$

$G_0$	$G_1$	$q = 3$ $q = 5$
1	3	5, 6, 9, 10, 12–15, 17, 19, 20–23, 26, 30 5, 6, 8–10, 12–15, 16, 17, 19–23, 25–28, 30
1	4	6, 10, 13, 15, 17, 18, 20, 22, 25, 26, 28, 29 6, 8, 10, 11, 13, 15–18, 20–22, 25–30
1	5	9, 12–15, 18–20, 22, 23, 26, 29, 30 2, 8, 9, 12–16, 18–23, 25, 26, 29, 30
1	6	2, 3, 10–12, 15, 17, 18, 23, 25, 26, 30 2, 3, 8, 10, 12, 14–19, 21, 23, 25–30
1	7	3, 4, 9, 10, 12–14, 17, 18, 21, 22, 25, 26, 28–30 2–4, 9, 10, 12–14, 17–22, 25, 26, 28–30
1	8	2, 3, 5, 10–12, 15, 18, 21, 25, 28–30 2–5, 10–12, 14–16, 18, 20–23, 25, 27, 28–30
1	9	4, 5, 11, 13, 14, 17, 18, 20, 21, 23, 25, 26 2, 4–6, 11–14, 16–18, 20, 21, 23, 25–28, 30
2	5	6, 10, 13, 15, 17, 18, 20, 22, 25, 26, 28, 29 6, 8, 10, 11, 13, 15–18, 20–22, 25–30
2	6	3, 5, 9, 10–13, 15, 17, 18, 20, 21, 23, 25, 26, 28–30 3, 5, 7, 9–13, 15–21, 23, 25–27, 28–30
2	7	4, 6, 10, 12, 13–15, 18, 20, 22, 23, 26, 28, 29 6, 10, 12–15, 17, 18, 20–23, 26–29
2	8	5, 7, 9, 11, 12, 17, 19, 20, 23, 25, 26, 29, 30 3, 5, 7, 9, 11–13, 15–17, 19–23, 25–27, 29, 30
2	9	4, 6, 10, 14, 15, 18, 21–23, 25, 26, 28, 30 3, 4, 6, 8, 10, 13–16, 18, 19, 21–23, 25–28, 30
continued on next page		

Table 7: Impossible values of $k$ in sequences with $A = 1, B = 1$ (continued from previous page)		
$G_0$	$G_1$	q=3 q=5
3	7	9, 12–15, 18–20, 22, 23, 26, 29, 30 2, 8, 9, 12–16, 18, 19–23, 25, 26, 29, 30
3	8	4, 6, 10, 12–15, 18, 20, 22, 23, 26, 28, 29 6, 10, 12–15, 17, 18, 20–23, 26–29
3	9	4, 5, 7, 10, 11, 13–15, 17–20, 23, 25, 26, 29, 30 2, 4, 5, 7, 8, 10, 11, 13–20, 22, 23, 25–30
4	9	2, 3, 10–12, 15, 17, 18, 23, 25, 26, 30 2, 3, 8, 10, 12, 14–19, 21, 23, 25–30
6	4	5, 7, 9, 11, 12, 17, 19, 20, 23, 25, 26, 29, 30 3, 5, 7, 9, 11–13, 15–17, 19–23, 25–27, 29, 30
6	5	3, 4, 9, 10, 12–14, 17, 18, 21, 22, 25, 26, 28–30 2, 4, 3, 9, 10, 12–14, 17–22, 25, 26, 28–30
7	3	2, 6, 9, 12, 14, 17, 18, 20–22, 25, 28–30 2, 5, 6, 8, 9, 12, 14, 16–19, 20–22, 25, 27–30
7	4	2, 6, 9, 12, 14, 17, 18, 20–22, 25, 28–30 2, 5, 6, 8, 9, 12, 14, 16–22, 25, 27–30
7	5	4, 6, 10, 14, 15, 18, 21–23, 25, 26, 28, 30 3, 4, 6, 8, 10, 13–16, 18, 19, 21–23, 25–28, 30
8	2	3, 5, 13, 15, 17–19, 21, 23, 25, 26, 28–30 3–5, 7, 9, 11, 13, 15–19, 21, 23, 25–30
8	3	2, 4, 6, 7, 9, 12, 15, 17, 19–23, 26, 28, 30 4, 6, 7, 9, 10, 12, 15–17, 19, 20–22, 23, 26–30
8	4	3, 5–7, 10, 11, 13, 15, 17–23, 25, 26, 29, 30 2, 3, 5–7, 9–11, 13–15, 17–23, 25–27, 29, 30
8	5	2, 4, 6, 7, 9, 12, 15, 17, 19–23, 26, 28, 30 4, 6, 7, 9, 10, 12, 15–17, 19–23, 26–30
8	6	3, 5, 13, 15, 17–19, 21, 23, 25, 26, 28–30 3–5, 7, 9, 11, 13, 15–19, 21, 23, 25–30
8	7	4, 5, 11, 13, 14, 17, 18, 20, 21, 23, 25, 26 2, 4–6, 11–14, 16–18, 20, 21, 23, 25–28, 30
9	1	2, 5–7, 12, 13, 15, 18–20, 23, 26, 28–30 2–7, 12–16, 18–20, 22, 23, 26–30
9	2	4–6, 10, 12, 14, 15, 17, 20, 25, 26, 28–30 3–6, 8, 10, 12, 14, 15, 17–22, 25–30
9	3	2, 4, 5, 7, 10, 11, 13, 14, 17–20, 22, 23, 25, 26, 28–30 2, 4, 5, 7, 8, 10, 11, 13, 14, 16–20, 22, 23, 25, 26, 28–30
9	4	3, 6, 7, 10–12, 15, 18, 20, 22, 25, 26, 29 2, 3, 6, 7, 8, 10–12, 15, 16, 18, 20–23, 25–29
9	5	3, 6, 7, 10–12, 15, 18, 20, 22, 25, 26, 29 2, 3, 6–8, 10–12, 15, 16, 18, 20–23, 25–29
continued on next page		



Table 7: Impossible values of $k$ in sequences with $A = 1$ , $B = 1$ (continued from previous page)		
$G_0$	$G_1$	$q=3$ $q=5$
9	6	2, 4, 5, 7, 10, 11, 13, 14, 17–20, 22, 23, 25, 26, 28–30 2, 4, 5, 7, 8, 10, 11, 13, 14, 16–20, 22, 23, 25, 26, 28–30
9	7	4–6, 10, 12, 14, 15, 17, 20, 25, 26, 28–30 3–6, 8, 10, 12, 14, 15, 17–22, 25–30
9	8	2, 5–7, 12, 13, 15, 18–20, 23, 26, 28–30 2–7, 12–16, 18–22, 23, 26–30

TABLE 8  
Values of  $q$ -power free constants  $2 \leq k \leq 30$  for which the equation  
 $G_n = kx^q$  has no solutions with  $A = 1$ ,  $B = 2$ , and  $q = 3, 5$

$G_0$	$G_1$	$q = 3$ $q = 5$
1	4	3, 5, 9–11, 13, 15, 17–23, 25, 28–30 2, 3, 5, 7, 9–13, 15–17, 19–22, 24, 25, 27, 29, 30
1	5	3, 6, 9, 11–15, 18–23, 25, 26, 28–30 3, 4, 6, 8–15, 18–27, 29, 30
1	6	2–5, 7, 9–12, 14, 15, 17–19, 21–23, 25, 26, 28–30 2–5, 7, 9, 10–19, 21–23, 25–30
1	7	5, 6, 10, 11, 15, 17–22, 25, 26, 28–30 4–6, 8, 10–22, 24–30
1	8	2–7, 9, 11–15, 17, 18, 20–23, 25, 29, 30 2–7, 9, 11–25, 27–30
1	9	2, 3, 5–7, 10, 13–16, 18–21, 23–26, 28, 30 2, 3, 5–8, 10, 12–28, 30
2	3	5, 9–12, 14, 15, 17, 19–22, 25, 26, 29, 30 5, 6, 8, 10–12, 15, 17–23, 25, 26, 28–30
2	5	3, 4, 6, 7, 10, 11, 13–15, 17, 18, 20–23, 25, 28–30 3, 4, 7, 10–12, 14–17, 18, 21–30
2	7	3–6, 9, 10, 12–15, 17–19, 21–23, 26, 28–30 3–6, 9, 10, 12–19, 20–24, 27–30
2	8	4–7, 10, 11, 13, 15, 17–23, 25, 26, 29, 30 5–7, 9–11, 13–15, 17–27, 29, 30
2	9	3–5, 7, 10–12, 14, 15, 18–23, 25, 26, 29, 30 3–8, 10–12, 14–23, 25–30
3	4	2, 5–7, 9, 11–15, 17, 19–23, 25, 26, 28–30
continued on next page		

Table 8: Impossible values of $k$ in sequences with $A = 1$ , $B = 2$ (continued from previous page)		
$G_0$	$G_1$	$q = 3$ $q = 5$
		2, 5–9, 11, 13–15, 17, 20–24, 25, 27–30
3	8	4–7, 10–13, 15, 17–19, 21–23, 25, 26, 29 2, 4–7, 9–13, 15–24, 26–29
4	2	3, 5–7, 9, 11–13, 15, 18, 19, 21–23, 25, 26, 28–30 3, 5–9, 11–13, 15–30
4	3	2, 5–7, 9, 10, 12–15, 19–23, 25, 26, 28–30 2, 5–10, 12–15, 18–30
4	5	2, 3, 6, 7, 9–12, 15, 17–22, 25, 26, 28–30 2, 3, 6–12, 14, 15, 17–19, 21, 22, 24–28, 30
4	7	2, 3, 5, 6, 9, 11, 13, 17–23, 26, 28, 30 2, 3, 5, 6, 8–14, 16, 17, 19–28, 30
4	9	2, 3, 5, 10–15, 18, 19, 21–23, 25, 26, 28, 29 3, 5–8, 10–14, 16, 18–23, 25–27, 29, 30
5	1	3, 4, 6, 7, 9, 10, 12, 14, 15, 17–21, 23, 26, 29, 30 3, 4, 6–10, 12, 14–30
5	2	3, 4, 6, 7, 10, 11, 13–15, 17, 18, 20–23, 25, 28–30 3, 4, 6–11, 13–15, 17–30
5	3	6, 7, 9–12, 14, 15, 17, 21–23, 25, 26, 28–30 4, 6–12, 14–18, 20–30
5	4	2, 3, 6, 7, 9–12, 15, 17–21, 23, 25, 26, 28–30 2, 3, 6–13, 15, 17–21, 23–30
5	6	3, 9–15, 17, 19–23, 25, 26, 29, 30 2–4, 7–15, 17–27, 29, 30
5	8	2, 3, 6, 7, 9–11, 13, 15, 17, 19–23, 25, 26, 28–30 2, 3, 6, 7, 9–16, 19–29
6	1	2–5, 7, 9–12, 14, 17–19, 21–23, 25, 26, 28–30 2–5, 7–12, 14, 16–30
6	2	5, 7, 9–13, 15, 17, 19–23, 25, 29, 30 5, 7–13, 15–17, 19–30
6	3	2, 4, 5, 7, 9–11, 13, 14, 17–20, 22, 23, 25, 26, 28, 29 2, 4, 5, 7–14, 16–20, 22–30
6	4	5, 9–12, 14, 15, 17, 19–22, 25, 26, 29, 30 2, 3, 5, 7–15, 17–23, 25–30
6	5	2, 3, 7, 9–14, 18–23, 25, 28–30 2–4, 7–15, 18–26, 28–30
6	7	2, 3, 5, 10–15, 17, 18, 20, 21, 23, 25, 26, 28–30 2–5, 8–15, 17, 18, 20–30
6	9	4, 5, 7, 10, 11, 13–15, 17, 19, 20, 22, 23, 25, 26, 28–30 2–5, 7, 8, 10–20, 22–26, 28–30
7	1	2, 6, 9–14, 18–23, 25, 26, 28–30
continued on next page		

Table 8: Impossible values of $k$ in sequences with $A = 1$ , $B = 2$ (continued from previous page)		
$G_0$	$G_1$	$q = 3$ $q = 5$
		2, 6, 8–14, 16, 18–30
7	2	3–6, 9–15, 17–19, 21–23, 25, 26, 28, 30 3–6, 8–15, 17–19, 21–30
7	3	4–6, 9–15, 18–22, 25, 28–30 4–6, 8–16, 18–22, 24–30
7	4	2, 3, 5, 6, 9–11, 13–15, 17, 19–22, 25, 28–30 2, 3, 5, 6, 8–17, 19–25, 27–30
7	5	2, 3, 6, 9–15, 17, 18, 21, 22, 25, 28, 30 2, 3, 6, 8–18, 20–28, 30
7	6	2, 3, 5, 10–15, 18, 19, 21–23, 25, 26, 28, 29 2–5, 8–15, 17–19, 21–30
7	8	2, 3, 5, 6, 9–15, 17–21, 23, 25, 28–30 2–6, 9–15, 17–21, 23–30
8	1	2–7, 9–15, 18, 20–23, 25, 26, 29, 30 2–7, 9–16, 18, 20–30
8	2	4–7, 9, 10, 12–15, 17, 19–21, 23, 25, 26, 28–30 4–7, 9–17, 19–21, 23–30
8	3	2, 5–7, 9, 11–15, 17, 18, 21–23, 26, 28–30 2, 4–7, 9–18, 20–24, 26–30
8	4	3, 6, 7, 9–15, 17–19, 21–23, 25, 26, 29, 30 3, 6, 7, 9–19, 21–27, 29, 30
8	5	2, 3, 6, 7, 9–11, 13–15, 17–20, 22, 23, 26, 28–30 2–4, 6, 7, 9–20, 22–30
8	6	2–5, 7, 9–15, 17–21, 23, 25, 26, 28, 30 2–5, 7, 9–21, 23–30
8	7	2, 3, 5, 6, 9–15, 17, 18, 20–22, 25, 26, 28–30 2–6, 9–15, 17–22, 24–30
9	1	2, 3, 5–7, 10–15, 17, 18, 20, 22, 23, 25, 26, 28–30 2, 3, 5, 6, 8, 10–18, 20, 22–30
9	2	4–7, 10–13, 15, 17–19, 21–23, 25, 26, 29 3–8, 10–19, 21–23, 25–30
9	3	2, 4, 5, 7, 10, 11, 13–15, 17–19, 20, 22, 23, 25, 26, 28, 29, 30 2, 4, 5, 7, 8, 10–20, 22, 23, 24–26, 28–30
9	4	2, 3, 5–7, 10–15, 17–19, 21, 23, 25, 26, 28, 29 2, 3, 5–8, 10–21, 23–29
9	5	3, 4, 6, 7, 10–15, 17–22, 25, 26, 28, 29 3, 4, 6–8, 10–22, 24–30
9	6	2, 4, 5, 7, 10, 11, 13–15, 17–23, 25, 26, 28–30 2–5, 7, 8, 10–23, 25–30
9	7	2, 6, 10–15, 17–23, 26
continued on next page		

Table 8: Impossible values of $k$ in sequences with $A = 1$ , $B = 2$ (continued from previous page)		
$G_0$	$G_1$	$q = 3$ $q = 5$
		2, 6, 8, 10–24, 26–30
9	8	2, 3, 5–7, 10–15, 17–20, 22, 23, 25, 28–30 2–7, 10–13, 15, 17–25, 27–30

TABLE 9  
Values of  $q$ -power free constants  $2 \leq k \leq 30$  for which the equation  
 $G_n = kx^q$  has no solutions with  $A = 2$ ,  $B = 1$ , and  $q = 3, 5$

$G_0$	$G_1$	$q = 3$ $q = 5$
1	1	2, 5, 6, 9–15, 18–23, 25, 26, 28–30 2, 4–6, 8–16, 18–30
1	3	2, 5, 6, 9–15, 18–23, 25, 26, 28–30 2, 4–6, 8–16, 18–30
1	4	5–7, 10–15, 17, 18, 20, 21, 23, 25, 26, 28–30 5–7, 10–18, 20, 21, 23–30
1	5	2, 6, 7, 9, 10, 12, 14, 15, 17–23, 25, 26, 28–30 2, 4, 6–10, 12, 14–26, 28–30
1	6	2, 3, 5, 9–12, 14, 15, 17, 19–23, 25, 26, 28–30 2, 3, 5, 8–12, 14–17, 19–25, 27–30
1	7	2, 3, 6, 10–14, 17–22, 25, 26, 28–30 2–4, 6, 8, 10–14, 16–22, 24–30
1	8	2, 3, 5, 7, 9, 10, 12–15, 18–22, 23, 25, 26, 29, 30 2–5, 7, 9, 10, 12–16, 18–27, 29, 30
1	9	2, 3, 5, 6, 10–12, 14, 15, 17, 18, 20–23, 25, 26, 28–30 2–6, 8, 10–12, 14–18, 20–30
2	3	5–7, 10–15, 17, 18, 20, 21, 23, 25, 26, 28–30 5–7, 10–18, 20, 21, 23–30
2	7	5, 6, 9, 10, 12–15, 17, 18, 20–23, 25, 28–30 5, 6, 8–10, 12–15, 17–25, 27–30
2	8	3, 5, 7, 9–15, 17, 19, 20–23, 25, 26, 28–30 3, 5, 7, 9–15, 17, 19–30
2	9	3, 6, 7, 10–15, 17–19, 22, 23, 25, 26, 28–30 3, 4, 6, 7, 10–19, 22–30
3	3	2, 5–7, 10, 12–15, 17–20, 22, 23, 25, 26, 28–30
continued on next page		

Table 9: Impossible values of $k$ in sequences with $A = 2$ , $B = 1$ (continued from previous page)		
$G_0$	$G_1$	$q = 3$ $q = 5$
		2, 4–8, 10–20, 22–30
3	4	5, 6, 9, 10, 12–15, 17, 18, 20–23, 25, 28–30 5, 6, 8–10, 12–15, 17–25, 27–30
3	5	2, 6, 7, 9, 10, 12, 14, 15, 17–23, 25, 26, 28–30 2, 4, 6–10, 12, 14–26, 28–30
3	9	2, 5–7, 10, 12–15, 17–20, 22, 23, 25, 26, 28–30 2, 4–8, 10–20, 22–30
4	3	2, 6, 11–13, 15, 17–22, 25, 26, 28–30 2, 6–9, 11–13, 15–22, 24–30
4	4	2, 3, 5–7, 9–11, 13–15, 17–20, 21–23, 25, 26, 29, 30 2, 3, 5–11, 13–27, 29, 30
4	5	2, 6, 11–13, 15, 17–22, 25, 26, 28–30 2, 6–9, 11–13, 15–22, 24–30
4	6	3, 5, 7, 9–15, 17, 19–23, 25, 26, 28–30 3, 5, 7, 9–15, 17, 19–30
4	7	2, 3, 5, 9–12, 14, 15, 17, 19–23, 25, 26, 28–30 2, 3, 5, 8–12, 14–17, 19–25, 27–30
5	3	2, 6, 9, 10, 12–15, 17, 18, 20–23, 26, 28–30 2, 4, 6, 8–10, 12–18, 20–24, 26–30
5	4	2, 3, 7, 9–12, 14, 15, 18, 19–21, 23, 25, 26, 28, 29 2, 3, 7–12, 14–16, 18–29
5	5	2, 3, 6, 7, 9–14, 17–23, 25, 26, 28–30 2–4, 6–14, 16–30
5	6	2, 3, 7, 9–12, 14, 15, 18–21, 23, 25, 26, 28, 29 2, 3, 7–12, 14–16, 18–29
5	7	2, 6, 9, 10, 12–15, 17, 18, 20–23, 26, 28–30 2, 4, 6, 8–10, 12–18, 20–24, 26–30
5	8	3, 6, 7, 10–15, 17–19, 22, 23, 25, 26, 28–30 3, 4, 6, 7, 10–19, 22–30
5	9	2, 3, 6, 10–14, 17–22, 25, 26, 28–30 2, 3, 4, 6, 8, 10–14, 16–19, 20–22, 24–30
6	2	3, 5, 7, 9, 11–15, 17–21, 23, 25, 28–30 3–5, 7–9, 11–21, 23–25, 27–30
6	3	2, 5, 7, 10, 11, 13–15, 17–23, 25, 26, 28–30 2, 4, 5, 7, 8, 10, 11, 13–23, 25, 26, 28–30
6	4	2, 3, 5, 7, 9–13, 15, 17–21, 23, 25, 26, 28–30 2, 3, 5, 7, 9–13, 15–21, 23–30
6	5	3, 9–15, 17–19, 21–23, 25, 26, 28–30 2–4, 8–15, 17–19, 21–30
6	6	2, 3, 5, 7, 9–15, 17, 19–21, 23, 25, 26, 28–30
continued on next page		

Table 9: Impossible values of $k$ in sequences with $A = 2$ , $B = 1$ (continued from previous page)		
$G_0$	$G_1$	$q = 3$ $q = 5$
		2-5, 7-17, 19-30
6	7	3, 9-15, 17-19, 21-23, 25, 26, 28-30 2-4, 8-15, 17-19, 21-30
6	8	2, 3, 5, 7, 9-13, 15, 17-21, 23, 25, 26, 28-30 2, 3, 5, 7, 9-13, 15-21, 23-30
6	9	2, 5, 7, 10, 11, 13-15, 17-23, 25, 26, 28-30 2, 4, 5, 7, 8, 10, 11, 13-23, 25, 26, 28-30
7	2	5, 6, 9, 10, 13-15, 17-23, 25, 26, 28-30 3-6, 8-10, 13-23, 25-30
7	3	2, 5, 6, 9, 10, 12, 14, 15, 17-23, 25, 26, 28, 30 2, 4, 5, 6, 8-10, 12, 14-28, 30
7	4	2, 3, 5, 6, 9, 11-14, 17-23, 26, 28-30 3, 5, 6, 8, 9, 11-14, 16-26, 28-30
7	5	2, 3, 6, 10-15, 18-23, 26, 28-30 2-4, 6, 8, 10-16, 18-24, 26-30
7	6	3, 5, 9-14, 15, 17, 18, 20-22, 25, 26, 28-30 2-5, 9-18, 20-22, 24-30
7	7	2, 3, 5, 6, 9-15, 17-20, 22, 23, 25, 26, 28-30 2-6, 8-20, 22-30
7	8	3, 5, 9-15, 17, 18, 20-22, 25, 26, 28-30 2-5, 9-18, 20-22, 24-30
7	9	2, 3, 6, 10-15, 18-23, 26, 28-30 2-4, 6, 8, 10-16, 18-24, 26-30
8	1	2, 3, 5-7, 9, 11-14, 17-20, 22, 23, 25, 26, 28-30 2, 4, 5, 6, 7, 9, 11-14, 16-20, 22-30
8	2	3, 5-7, 9-11, 13, 15, 17-23, 25, 28-30 3-7, 9-11, 13, 15-25, 27-30
8	3	2, 5-7, 9-12, 15, 17-23, 25, 26, 28-30 2, 4-7, 9-12, 15-30
8	4	3, 5-7, 9, 10, 13-15, 17-22, 25, 26, 28-30 2, 3, 5-7, 9-11, 13-15, 17-30
8	5	2, 3, 6, 7, 9, 10, 12-15, 17, 19-23, 25, 26, 28, 29 2-4, 6, 7, 9, 10, 12-17, 19-29
8	6	2, 3, 5, 7, 9, 11-13, 15, 17, 19, 21-23, 25, 26, 29, 30 2, 3, 4, 7, 9, 11-19, 21-27, 29, 30
8	7	2, 3, 5, 6, 10-15, 17-21, 23, 25, 28-30 2-6, 10-21, 23-25, 27-30
8	8	2, 5, 6, 9-15, 18-23, 25, 26, 28-30 2-7, 9-23, 25-30
8	9	2, 3, 5, 6, 10-15, 17-21, 23, 25, 28-30
continued on next page		

Table 9: Impossible values of $k$ in sequences with $A = 2$ , $B = 1$ (continued from previous page)		
$G_0$	$G_1$	$q = 3$ $q = 5$
		2-6, 10-21, 23-25, 27-30
9	1	2, 3, 5-7, 10, 12-15, 18-22, 25, 26, 28-30 2-8, 10, 12-16, 18-22, 24-30
9	2	3, 5-7, 10-12, 14, 15, 17-23, 25, 26, 29, 30 3-8, 10-12, 14, 15, 17-27, 29, 30
9	3	2, 5-7, 10-14, 17-23, 25, 26, 28-30 2, 4-8, 10-14, 16-30
9	4	2, 3, 5-7, 10, 12, 13, 15, 18-23, 25, 26, 29, 30 2, 3, 5, 6, 8, 10-13, 15, 16, 18-30
9	5	2, 3, 6, 7, 10-12, 14, 15, 17, 18, 20-23, 25, 26, 28-30 2-4, 6-8, 10-12, 14-18, 20-30
9	6	2, 3, 5, 10, 11, 13-15, 17-20, 22, 23, 25, 26, 28-30 2-5, 7, 8, 10, 11, 13-20, 22-30
9	7	2, 3, 5, 6, 10, 12-15, 17-22, 25, 26, 28-30 2-6, 8, 10, 12-22, 24-30
9	8	2, 3, 5-7, 11-15, 17-23, 26, 28, 30 2-7, 11-24, 26-28, 30
9	9	2, 3, 5-7, 10-15, 17-23, 25, 26, 28-30 2-8, 10-26, 28-30

TABLE 10  
Values of  $q$ -power free constants  $2 \leq k \leq 30$  for which the equation  
 $G_n = kx^q$  has no solutions with  $A = 1$ ,  $B = 3$ , and  $q = 3, 5$

$G_0$	$G_1$	$q = 3$	$q = 5$
1	2	29	3, 4, 6-8, 10, 12-14, 16-25, 27-30
1	3	22	2, 4, 5, 7-14, 16, 18-26, 28-30
1	5	2, 6	2-4, 6, 7, 9-22, 24-26, 28-30
1	6	29	2-5, 7, 8, 10-26, 28-30
1	7	30	3-6, 8, 9, 11-30
1	8	29, 30	2-7, 9, 10, 12-27, 29, 30
1	9	11, 20, 30	2-5, 7, 8, 10, 11, 13-30
2	3	25	4-8, 10-17, 19-21, 22, 24-30
2	4	25	3, 5-9, 11-17, 19-21, 23-29
2	6	13, 29	3-5, 7-11, 13-29
2	7	6, 10	3-6, 8-12, 14-26, 28-30
continued on next page			

Table 10: Impossible values of $k$ in sequences with $A = 1$ , $B = 3$ (continued from previous page)			
$G_0$	$G_1$	$q = 3$	$q = 5$
2	9	5, 13	3–8, 10–14, 16–26, 28–30
3	1	4, 9, 15, 22, 23	2, 4–9, 11, 12, 14–30
3	2	5, 22, 25, 26	4–10, 12–16, 18–30
3	4	10, 19, 26	5–12, 14–24, 26–30
3	5	12, 19, 26	2, 4, 6–8, 10–13, 15–24, 26–28, 30
3	7		4–6, 8–12, 14, 15, 17–23, 25–30
3	8	2, 5, 15, 22, 30	2, 4–7, 9–16, 18–30
4	1	6, 12, 18	2, 3, 5–12, 14, 15, 17–30
4	2	23	3, 5–13, 15–19, 21–30
4	3	5, 6, 11, 18	2, 5–14, 16–23, 25–30
4	5	6, 10, 12, 14, 23	2, 3, 6–16, 18–30
4	6		2, 3, 5, 7–17, 19–30
4	8	25	2, 3, 5–7, 9, 10–19, 21–30
4	9	13	2, 3, 5, 7, 8, 10, 12–20, 22–30
5	1	3, 30	2–4, 6–15, 17, 18, 20–30
5	2	3, 29	3, 4, 6–16, 18–22, 24–30
5	3	22	2, 4, 6–17, 19–26, 28–30
5	4	3, 10, 25	2, 3, 6–17, 18, 20–30
5	6	4, 18, 30	2–4, 7–20, 22–30
5	7	11, 26	2–4, 6, 8–21, 23–30
5	9	6, 10, 11, 15, 26	2, 4, 6–8, 10–23, 25–28, 30
6	1	3, 4, 13, 23, 25	2–5, 7–18, 20, 21, 23–30
6	2	18, 19, 29, 30	3–5, 7–19, 21–25, 27–30
6	3	5, 10, 23	2, 4, 5, 7–20, 22–29
6	4	2, 10, 17, 19	2, 3, 5, 7–21, 23–30
6	5	12, 25, 26	2–4, 7–22, 24–30
6	7	10, 22	2–5, 8–24, 26–30
6	8	17, 20, 23	2, 3, 5, 7, 9–25, 27–30
7	1	9, 18, 26	4–6, 8, 9–21, 23, 24, 26–30
7	2	13, 30	3–6, 8–22, 24–28, 30
7	3	5, 19	2, 4–6, 8–23, 25–30
7	4		2, 3, 5, 6, 8–24, 26–30
7	5	13	2–4, 6, 8–25, 27–30
7	6	4, 15, 22, 26	2–5, 8–26, 28–30
7	8	23	2, 3, 5, 6, 9–28, 30
7	9		2–6, 8, 10–29
8	1	4, 5, 19, 26	2–7, 9–24, 26, 27, 29, 30
8	2	3, 12, 13	3–7, 9–25, 27–30
8	3	13, 17	2, 4–6, 7, 9–17, 19–26, 28–30
8	4	15	2, 3, 5–7, 9–27, 29, 30
continued on next page			



Table 10: Impossible values of  $k$  in sequences with  $A = 1$ ,  $B = 3$   
(continued from previous page)

$G_0$	$G_1$	$q = 3$	$q = 5$
8	5	4, 20, 30	2, 4, 7, 9–28, 30
8	6	10, 12, 22, 25	2–5, 7, 10–29
8	7	3, 5, 6, 10, 15, 18, 20, 21	2–6, 9–22, 24–30
8	9	10, 22	2–7, 10–30
9	1	23	2–8, 10–27, 29, 30
9	2	17, 18, 19, 20, 25	3–8, 10–28, 30
9	3	5, 11, 12, 22	4–8, 10–29
9	4	18, 22, 29	2, 3, 5–8, 10–26, 28–30
9	5	2, 15, 26, 29, 30	2–4, 6–8, 10–30
9	6	3, 5, 13, 15, 23	2–5, 7, 8, 10–30
9	7	6	2–6, 8, 10–30
9	8	10, 11, 22, 23, 30	2–7, 10–30

TABLE 11

Values of  $q$ -power free constants  $2 \leq k \leq 30$  for which the equation  
 $G_n = kx^q$  has no solutions with  $A = 2$ ,  $B = 2$ , and  $q = 3, 5$

$G_0$	$G_1$	$q = 3$ $q = 5$
1	1	2, 3, 5–7, 9, 11–13, 15, 17, 18, 20–23, 25, 29, 30 2, 3, 5–9, 11–15, 17–27, 29, 30
1	5	2–4, 6, 7, 9–11, 13, 15, 17, 19–23, 26, 28–30 2–4, 6–11, 13–30
1	6	3, 7, 9–11, 13, 15, 17–19, 21–23, 25, 28–30 3–5, 7–13, 15–30
1	7	3–6, 9–15, 17–19, 21–23, 25, 28–30 3–6, 9–15, 17–28, 30
1	8	2, 5, 7, 9–15, 17, 19, 21–23, 25, 26, 28–30 2, 5–7, 9–11, 13–17, 19–30
1	9	2, 4–7, 10–15, 17–19, 21–23, 25, 26, 29, 30 2, 4, 5–8, 10–19, 21–30
2	2	3–6, 9, 10–15, 17, 18, 21–23, 25, 26, 29, 30 3–7, 9–12, 14–19, 21–30
2	3	5–7, 11, 13, 15, 17–19, 21–23, 25, 28–30 4–9, 11–15, 17–25, 27–30
2	5	3, 6, 9–12, 15, 17–23, 25, 26, 28–30

continued on next page

Table 11: Impossible values of $k$ in sequences with $A = 2$ , $B = 2$ <i>continued from previous page</i>		
$G_0$	$G_1$	$q = 3$ $q = 5$
		3, 4, 6, 7, 9, 10–13, 15, 17–19, 21–27, 29, 30
2	7	3, 5, 6, 9, 11, 13–15, 19–22, 25, 26, 28–30 3–6, 8–15, 17, 19–30
2	8	3–6, 9–15, 17, 18, 21–23, 25, 26, 29, 30 3–7, 9–12, 14–19, 21–30
2	9	3–7, 10, 11, 13–15, 17–19, 23, 25, 26, 29, 30 3–8, 10–21, 23–30
3	2	4–7, 9, 11–15, 17–22, 25, 26, 29, 30 4–9, 11–23, 25–30
3	3	2, 4–7, 9–11, 13–15, 17–23, 25, 26, 28, 29 2, 4–11, 13–29
3	4	2, 5–7, 9–13, 15, 17–19, 21–23, 25, 26, 28–30 2, 5–13, 15–30
3	5	7, 9–15, 17, 19–23, 25, 28–30 4, 7–15, 17–26, 28–30
3	7	5, 9, 11, 12, 14, 15, 17, 19, 21–23, 25, 26, 28–30 2, 4–6, 8–15, 17–19, 21–23, 25–30
4	1	2, 3, 5–7, 9, 11–15, 17–21, 23, 25, 26, 29, 30 3, 5–9, 11–30
4	2	6, 7, 11, 13–15, 17–23, 25, 26, 29, 30 6–8, 10, 11, 13–27, 29, 30
4	3	2, 5–7, 10, 11, 13, 15, 17–19, 21–23, 25, 26, 28–30 2, 5–7, 9–13, 15–30
4	4	3, 6, 9–12, 15, 17–23, 25, 26, 28–30 3, 6, 8–12, 14, 15, 17–25, 27–30
4	5	3, 7, 9–11, 13–15, 17, 19–23, 25, 26, 29, 30 3, 7–17, 19–30
4	6	7, 9–15, 17, 19, 21–23, 25, 26, 29, 30 2, 7–19, 21–30
4	7	2, 3, 5, 6, 9–15, 17–19, 21, 23, 25, 26, 28–30 2, 3, 6, 8–15, 17–21, 23–30
4	9	5–7, 11, 13–15, 17–20, 22, 23, 25, 28, 30 3, 5, 7, 8, 10–15, 17–25, 27–30
5	1	2–4, 6, 9–11, 13–15, 17–23, 25, 28–30 2–4, 6, 8–11, 13–25, 27–30
5	2	3, 6, 7, 9–13, 15, 17–23, 25, 26, 28, 30 3, 6–13, 15, 17–30
5	3	7, 9–15, 17–19, 21–23, 25, 26, 29, 30 2, 4, 7–15, 17–19, 21–24, 26–30
5	4	2, 6, 9–15, 17, 19–23, 25, 26, 28–30
<i>continued on next page</i>		

Table 11: Impossible values of $k$ in sequences with $A = 2$ , $B = 2$ <i>continued from previous page</i>		
$G_0$	$G_1$	$q = 3$ $q = 5$
		2, 6, 8–17, 19–30
5	5	2–4, 6, 7, 9–15, 17–19, 21–23, 25, 26, 28–30 2–4, 6–19, 21–30
5	6	3, 4, 9–15, 17–21, 23, 25, 26, 28–30 3, 4, 7, 8, 10–21, 23–26, 27–30
5	7	6, 9–11, 13–15, 17–19, 21–23, 25, 26, 28–30 2, 3, 6, 8–23, 25–30
5	8	2, 3, 6, 7, 9–15, 17–23, 25, 29, 30 2, 3, 6, 7, 9–15, 17–25, 27–30
5	9	2, 6, 7, 10–15, 17–23, 25, 29, 30 2, 4, 6–8, 10–15, 17–27, 29, 30
6	1	2, 4, 5, 7, 9, 10, 12, 13, 15, 17–23, 25, 26, 28, 29 2–5, 7–13, 15–29
6	2	3, 4, 7, 9–11, 14, 15, 17–23, 25, 26, 28–30 3, 4, 7, 9–15, 17–23, 25, 26, 28–30
6	3	2, 5, 7, 9–11, 13, 14, 17, 19–23, 25, 26, 28–30 2, 4, 5, 7–17, 19–30
6	4	2, 3, 5, 10–15, 18, 19, 21–23, 25, 26, 28–30 2, 3, 5, 8, 10–16, 18, 19, 21–30
6	5	3, 4, 7, 9–15, 17, 18, 20, 21, 23, 25, 26, 29, 30 2–4, 7–21, 23–30
6	6	2, 4, 5, 7, 9–15, 17–20, 22, 23, 25, 26, 28–30 2, 4, 5, 7–23, 25–30
6	7	3, 5, 9–15, 17–19, 21, 22, 25, 28–30 2–5, 8–25, 27–30
6	8	3, 4, 7, 10–15, 17–23, 26, 29, 30 3, 4, 7, 9–16, 18–27, 29, 30
6	9	2, 4, 5, 7, 10, 11, 13–15, 17–23, 25, 26, 28, 29 2–5, 7, 8, 10–29
7	1	3–6, 9, 11–15, 17–23, 25, 26, 28–30 2–6, 8, 9, 11–15, 17–22, 24–30
7	2	3, 4, 9–12, 14, 15, 17, 19–23, 25, 26, 28–30 3–5, 8–17, 19–30
7	3	2, 4–6, 10–15, 17–19, 21–23, 25, 26, 28–30 2, 4, 5, 6, 8, 10–19, 21–30
7	4	2, 3, 6, 9, 10, 12–15, 17–21, 23, 25, 26, 28–30 2, 3, 6, 8–10, 12–21, 23–30
7	5	2, 4, 6, 9–15, 17, 18, 20–23, 25, 26, 28–30 2–4, 6, 9–23, 25–30
7	6	2, 3, 5, 9–15, 17, 19–23, 25, 28–30
<i>continued on next page</i>		

Table 11: Impossible values of $k$ in sequences with $A = 2$ , $B = 2$ <i>continued from previous page</i>		
$G_0$	$G_1$	$q = 3$ $q = 5$
		3, 5, 8–17, 19–25, 27–30
7	7	2–6, 9–15, 17–23, 25, 26, 29, 30 2–6, 8–27, 29, 30
7	8	2, 4–6, 10–15, 17–23, 25, 26, 28, 29 2, 4–6, 9–17, 19–29
7	9	2, 3, 5, 10–15, 17, 18, 21–23, 25, 26, 28–30 2–5, 8, 10–18, 20–30
8	1	2–7, 9–11, 13, 15, 17, 19, 20, 22, 23, 25, 26, 28–30 2–5, 7, 9–17, 19, 20, 22–30
8	2	3–6, 9, 10, 12–15, 17–19, 21–23, 25, 26, 28–30 3, 5, 6, 9, 10, 12–19, 21–30
8	3	2, 4–7, 9–15, 17, 20, 21, 23, 25, 26, 28–30 2, 4–7, 9–18, 20, 21, 23–30
8	4	5, 9, 11, 12, 14, 15, 17, 19, 21–23, 25, 26, 28–30 2, 3, 7, 9, 11, 12, 14–17, 19–23, 25–30
8	5	2–4, 6, 7, 9–15, 18, 20, 21, 23, 25, 28–30 2–4, 6, 7, 9–16, 18–25, 27–30
8	6	4, 7, 10–15, 17, 19–23, 25, 26, 29, 30 2–4, 7, 10–15, 17–27, 29, 30
8	7	2–6, 9–14, 17–23, 25, 28, 29 2–6, 9–14, 16–29
8	8	2, 3, 5–7, 9, 11–13, 15, 17, 18, 20–23, 25, 29, 30 2, 3, 5, 6, 9, 11–13, 15–18, 20–25, 27–30
8	9	2–7, 10–12, 14, 15, 17–23, 25, 29 2–7, 10–12, 14–30
9	1	2–7, 10–12, 14, 15, 17–19, 21–23, 25, 26, 28–30 2–8, 10–12, 14–19, 21–30
9	2	3, 5, 7, 10–15, 17–21, 23, 25, 26, 28–30 3–7, 10–21, 23–30
9	3	4–7, 10, 11, 13–15, 17, 19–23, 25, 26, 28–30 2, 4–8, 10, 11, 13–23, 25–30
9	4	2, 3, 5, 6, 10–14, 17–23, 25, 28–30 2, 3, 5, 6, 8, 10–14, 16–25, 27–30
9	5	2–4, 6, 7, 10, 12–15, 17–23, 25, 26, 29, 30 2–4, 6–8, 10, 12–27, 29, 30
9	6	2, 3, 5, 10–15, 17–23, 25, 26, 28, 29 2–5, 7, 8, 10–29
9	7	2, 3, 5, 6, 11–15, 17–23, 25, 26, 28–30 2–6, 8, 11–30
9	8	2–4, 6, 7, 11, 13–15, 17–23, 25, 26, 28–30
<i>continued on next page</i>		

Table 11: Impossible values of $k$ in sequences with $A = 2$ , $B = 2$ <i>continued from previous page</i>		
$G_0$	$G_1$	$q = 3$ $q = 5$
		2-4, 6, 7, 10, 11, 13-19, 21-30
9	9	2-7, 10-15, 17-23, 25, 26, 28-30 2-8, 10-30

TABLE 12  
Values of  $q$ -power free constants  $2 \leq k \leq 30$  for which the equation  
 $G_n = kx^q$  has no solutions with  $A = 3$ ,  $B = 1$ , and  $q = 3, 5$

$G_0$	$G_1$	$q = 3$ $q = 5$
1	1	3, 5, 6, 9-12, 14, 15, 17-22, 25, 26, 28-30 3, 5, 6, 8-12, 14-22, 24-30
1	2	3, 5, 6, 9-12, 14, 15, 17-22, 25, 26, 28-30 3, 5, 6, 8-12, 14-22, 24-30
1	5	3, 4, 6, 9-15, 18-23, 25, 26, 28-30 3, 4, 6-15, 18-30
1	6	2, 4, 5, 7, 9-15, 17, 18, 20-23, 25, 28-30 2, 4, 5, 7, 9-18, 20-26, 28-30
1	7	2, 3, 5, 6, 9, 10, 12-15, 17-21, 23, 26, 28-30 2, 3, 5, 6, 8-10, 12-21, 23-30
1	8	2, 3, 4, 6, 7, 9-13, 15, 17-23, 26, 28-30 2-4, 6, 7, 9-13, 15, 17-24, 26-30
1	9	2-5, 7, 10-15, 18-22, 25, 26, 29, 30 2-5, 7, 8, 10-16, 18-27, 29, 30
2	2	3, 5, 6, 7, 9-13, 15, 17, 18, 20-23, 25, 28-30 3, 5-7, 9-13, 15-25, 27-30
2	3	4-7, 9, 10, 12-15, 17-23, 25, 26, 28-30 4-9, 10, 12-30
2	4	3, 5-7, 9-13, 15, 17, 18, 20-23, 25, 28-30 3, 5-7, 9-13, 15-25, 27-30
2	5	3, 4, 6, 9-15, 18-23, 25, 26, 28-30 3, 4, 6-15, 18-30
2	9	4-6, 10, 11, 13-15, 17-23, 25, 26, 28, 30 3, 4, 6-9, 10-15, 18-30
3	3	2, 4, 5, 7, 9-11, 13-15, 17-20, 22, 23, 25, 26, 28-30
<i>continued on next page</i>		

Table 12: Impossible values of $k$ in sequences with $A = 3, B = 1$ <i>continued from previous page</i>		
$G_0$	$G_1$	$q = 3$ $q = 5$
		2, 4, 5, 7–11, 13–20, 22–30
3	4	2, 7, 9–14, 17, 19–23, 25, 26, 28–30 2, 6–14, 16, 17, 19–30
3	5	2, 7, 9–14, 17, 19–23, 25, 26, 28–30 2, 6–14, 16, 17, 19–30
3	6	2, 4, 5, 7, 9–11, 13–15, 17–20, 22, 23, 25, 26, 28–30 2, 4, 5, 7–11, 13–20, 22–30
3	7	4–6, 10, 11, 13–15, 17–23, 25, 26, 28, 30 4–6, 8, 10–23, 25–28, 30
3	8	2, 4, 5, 7, 9–15, 17, 18, 20–23, 25, 28–30 2, 4, 5, 7, 9–18, 20–26, 28–30
4	2	3, 5–7, 9, 11–13, 15, 17–23, 25, 26, 28–30 3, 5–9, 11–30
4	3	2, 5–7, 10–12, 14, 15, 18–23, 25, 26, 28–30 2, 5–8, 10–12, 14–30
4	4	3, 5–7, 9–15, 17–23, 25, 26, 29, 30 2, 3, 5–7, 9–15, 17–27, 29, 30
4	5	2, 3, 6, 9–15, 17, 18, 20–23, 26, 28–30 2, 3, 6, 8–18, 20–24, 26–30
4	6	2, 3, 5, 7, 10–15, 17–21, 23, 25, 26, 28–30 2, 3, 5, 7–21, 23–30
4	7	2, 3, 6, 9–15, 17, 18, 20–23, 26, 28–30 2, 3, 6, 8–18, 20–24, 26–30
4	8	3, 5–7, 9–15, 17–23, 25, 26, 29, 30 2, 3, 5–7, 9–15, 17–27, 29, 30
4	9	2, 5–7, 10–12, 14, 15, 18–23, 25, 26, 28–30 2, 5–8, 10–12, 14–30
5	2	3, 4, 6, 7, 9, 10, 12, 14, 15, 17–23, 25, 26, 28–30 3, 4, 6–10, 12, 14–30
5	3	2, 4, 6, 7, 9–11, 15, 17–23, 25, 26, 28–30 2, 4, 6–11, 13, 15–30
5	4	2, 3, 6, 7, 9, 10, 12–15, 18–23, 25, 26, 28–30 2, 3, 6–10, 12–16, 18–30
5	5	2–4, 6, 7, 9, 11–15, 17–19, 21–23, 25, 26, 28–30 2–4, 6–9, 11–19, 21–30
5	6	2, 3, 7, 10–15, 17–22, 25, 26, 28–30 2–4, 7, 8, 10–22, 24–30
5	7	2–4, 6, 9–15, 17–23, 25, 28, 30 2–4, 6, 9–25, 27, 28, 30
5	8	2–4, 6, 9–15, 17–23, 25, 28, 30
<i>continued on next page</i>		

Table 12: Impossible values of $k$ in sequences with $A = 3$ , $B = 1$ <i>continued from previous page</i>		
$G_0$	$G_1$	$q = 3$ $q = 5$
		2, 3, 4, 6, 9–25, 27, 28, 30
5	9	2, 3, 7, 10–15, 17–22, 25, 26, 28–30 2–4, 7, 8, 10–22, 24–30
6	2	3–5, 7, 9–11, 13–15, 17–23, 25, 26, 28–30 3–5, 7–11, 14, 15, 17–30
6	3	2, 4, 5, 7, 9–14, 17–20, 22, 23, 25, 26, 28–30 2, 4, 5, 7–14, 16–30
6	4	2, 5, 7, 9–13, 15, 17, 19–23, 25, 26, 28–30 2, 3, 5, 7–13, 15–17, 19–30
6	5	2–4, 7, 9–12, 14, 15, 17–20, 22, 23, 25, 26, 28–30 2–4, 7–12, 14–20, 22–30
6	6	2, 4, 5, 7, 9–11, 13–15, 17–23, 25, 26, 28–30 2–5, 7–11, 13–23, 25–30
6	7	3–5, 9, 10, 12–15, 17–23, 25, 26, 28–30 2, 3, 5, 8, 10, 12–26, 28–30
6	8	2–5, 7, 9, 11, 13–15, 17–23, 25, 26, 28, 29 2–5, 7, 9, 11–29
6	9	2, 3, 5, 7, 10–15, 17–23, 25, 26, 28–30 2–5, 7, 8, 10–30
7	1	2–6, 9, 11–15, 17–19, 21–23, 25, 26, 28–30 2–6, 8, 9, 11–19, 21–30
7	2	3–6, 9–12, 14, 15, 18, 20–23, 25, 26, 28–30 3–6, 8, 9–12, 14–18, 20–30
7	3	4–6, 9–15, 17, 19–23, 25, 26, 28–30 2, 4–6, 8–15, 17, 19–30
7	4	2, 3, 5, 6, 9–15, 18, 20, 21, 23, 25, 26, 28–30 2, 3, 5, 6, 8–16, 18, 20–30
7	5	3, 4, 6, 9–15, 17–21, 23, 25, 26, 28–30 2–4, 6, 8–15, 17–21, 23–30
7	6	2, 4, 5, 9–14, 17–23, 26, 28–30 2, 3, 5, 8–14, 16–24, 26–30
7	7	2–6, 9–13, 15, 17–23, 25, 26, 29, 30 2–6, 8–13, 15–27, 29, 30
7	8	2–6, 9–12, 14, 15, 17–23, 25, 26, 28–30 2–6, 9–12, 14–30
7	9	2–6, 10, 11, 13–15, 17–23, 25, 26, 28–30 2–6, 8, 10, 11, 13–30
8	1	2–7, 9, 10, 12–15, 17–22, 25, 26, 28–30 2–7, 9, 10, 12–22, 24–30
8	2	3–7, 9–13, 15, 17–21, 23, 25, 26, 28–30
<i>continued on next page</i>		

Table 12: Impossible values of $k$ in sequences with $A = 3$ , $B = 1$ continued from previous page		
$G_0$	$G_1$	$q = 3$ $q = 5$
		3–7, 9–13, 15–21, 23–30
8	3	4–7, 9–15, 18–20, 22, 23, 25, 26, 28–30 2, 4–6, 7, 9–16, 18–20, 22–30
8	4	2, 3, 5–7, 9, 10–15, 17–19, 21–23, 25, 26, 29, 30 3, 5, 6, 9–19, 21–30
8	5	2–4, 6, 7, 9–13, 15, 17, 18, 20–22, 25, 26, 29, 30 2–4, 6, 7, 9–18, 20–22, 24–30
8	6	2–5, 7, 9–15, 17, 19–23, 25, 28–30 2–5, 7, 9–17, 19–25, 27–30
8	7	2–6, 9–15, 18–23, 25, 26, 28, 30 2–6, 9–16, 18–28, 30
8	8	3, 5, 6, 9–12, 14, 15, 17–22, 25, 26, 28–30 2–7, 9–15, 17, 18, 20–30
8	9	2–7, 10–14, 17–23, 25, 26, 28–30 2–7, 10–14, 16–30
9	1	2–7, 10, 11, 13–15, 17–23, 25, 28–30 2–8, 10, 11, 13–25, 27–30
9	2	3–7, 10–14, 17–23, 26, 28–30 3–6, 8, 10–14, 16–24, 26–30
9	3	2, 4–6, 10–15, 17, 19–23, 25, 26, 28–30 2, 4–8, 10–17, 19–23, 25–30
9	4	2, 3, 5–7, 10–15, 17–20, 22, 25, 26, 28–30 2, 3, 5–8, 10–20, 22, 24–30
9	5	2, 6, 10–15, 17–21, 23, 25, 26, 28–30 2–4, 6–8, 10–21, 23, 25–30
9	6	2–5, 7, 10–15, 17–20, 22, 23, 25, 26, 28, 30 2–5, 7, 8, 10–20, 22–26, 28–30
9	7	2–5, 10–15, 17–19, 21–23, 25, 26, 28, 29 2–6, 8, 10–19, 21–29
9	8	2–7, 10–15, 17, 18, 20–23, 25, 26, 28–30 2–7, 10–18, 20–30
9	9	2–7, 10–15, 17, 19–23, 25, 26, 28–30 2–8, 10–17, 19–30

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